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The Convergence of Padé Approximants to Functions with Branch Points

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Padé approximants are a natural generalization of Taylor polynomials; however instead of polynomials now rational functions are used for the development of a given function. In this article the convergence in capacity of Padé approximants $[m/n]$ with $m+n\to\infty$, $m/n\to 1$, is investigated. Two types of assumptions are considered: In the first case the function f to be approximated has to have all its singularities in a compact set $E\subseteq\mathbb{C}$ of capacity zero (the function may be multi-valued in $\overline{\mathbb{C}}\backslash E$). In the second case the function f has to be analytic in a domain possessing a certain symmetry property (this notion is defined and discussed below). It is shown that close-to-diagonal sequences of Padé approximants $[m/n]$ converge to f in capacity in a domain D that can be determined in various ways. In the case of the first type of assumptions the domain D is determined by the minimality of the capacity of the complement of D, in the second case the domain D is determined by a symmetry property. The rate of convergence is determined, and it is shown that this rate is best possible for convergence in capacity. In addition to the convergence results the asymptotic distribution of zeros and poles of the approximants is studied.

1. MAIN RESULTS

Functions

$$f(z)=f_0+\frac{f_1}{z}+\frac{f_2}{z^2}+\cdots \tag{1.1}$$

analytic in a neighborhood of infinity will be approximated by Padé approximants. In a first group of results rather strong assumptions are made about the singularities of the function f to be approximated (Assumption 1.1). In a second group (Definition 1.3 and Theorem 1.7) a different type of assumption is used, which turns out to be more general.

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Assumption 1.1. The function f is assumed to be analytic at infinity and to have all its singularities in a compact set $E \subseteq \overline{\mathbb{C}}$ with $\text{cap}(E)=0$, i.e. f has analytical continuations along any path in $\overline{\mathbb{C}}\backslash E$ starting at infinity; the continuation may be multiple-valued.

By $\text{cap}(\cdot)$ we denote the (logarithmic) capacity of a bounded (measurable) set in $\mathbb{C}$ (for a definition see [28, Apprendix I] or any other book on potential theory). The notion of capacity zero can be extended to subsets of the whole Riemann sphere $\overline{\mathbb{C}}$ by Möbius transforms, while positive capacity is defined only for sets bounded in $\mathbb{C}$.

DEFINITION 1.1. The *Padé approximant* $[m/n]$ of degree $m, n \in \mathbb{N}$ to the function f developed at infinity is defined as the rational function

$$[m/n](z) = \frac{p_{mn}(1/z)}{q_{mn}(1/z)}, \tag{1.2}$$

where the pair of *Padé polynomials* (p_{mn}, q_{mn}), $p_{mn} \in \mathscr{P}_m$, $q_{mn} \in \mathscr{P}_n$, $q_{mn} \not\equiv 0$, has to satisfy

$$q_{mn}\left(\frac{1}{z}\right) f(z) - p_{mn}\left(\frac{1}{z}\right) = O(z^{-m-n-1}) \qquad \text{as} \quad z \to \infty. \tag{1.3}$$

By $O(\cdot)$ we denote Landau's big "oh" and by $\mathscr{P}_n$ the set of all complex polynomials of degree not greater than n. The Padé approximant $[m/n]$ is uniquely determined by (1.3). This, however, is not the case for the Padé polynomials p_{mn} and q_{mn}. They can always be multiplied by a non-zero constant, but there may exist more essential non-uniqueness (cf. [17, Chap. V] or [1, Chap. I]). In what is called the normal case, the Padé approximants $[m/n]$ have a contact with f at infinity of order $m+n+1$. In general, this contact can be larger and also smaller than $m+n+1$. The Padé approximants (1.2) can be considered as the rational analogue of Taylor polynomials to a function f developed at infinity.

Since any continuum is of positive capacity, in Assumption 1.1 the set E cannot contain any continuum. Therefore if the function f has branch points, then different branch points must lie on different components of E. Hence, the analytic continuation of f is necessarily multivalued in $\mathbb{C}\backslash E$ *if* f has branch points.

As the point of departure for the discussion of the main results we take the

NUTTALL–POMMERENKE THEOREM [13, 18]. *Let f satisfy Assumption 1.1 and assume that f is single-valued in $\mathbb{C}\backslash E$. Then for any $\varepsilon > 0$, $0 < \lambda < 1$, and every compact set $V \subseteq \mathbb{C}\backslash E$, we have*

$$\lim_{m, n \to \infty} \operatorname{cap}\{z \in V \mid |(f - [m/n])(z)| > \varepsilon^{m+n}\} = 0, \tag{1.4}$$

where the sequence $(m, n) \in \mathbb{N}^2$ is supposed to satisfy

$$\lambda n \leqslant m \leqslant \frac{n}{\lambda} \quad \text{as} \quad m, n \to \infty. \tag{1.5}$$

Motivated by the Nuttall–Pommerenke Theorem, *convergence in capacity* is defined in analogy to convergence in measure in the following way:

DEFINITION 1.2. A sequence of functions f_n, $n = 1, 2, ...,$ is said to converge in capacity to f in the domain $D \subseteq \overline{\mathbb{C}}$ if for every $\varepsilon > 0$ and every compact set $V \subseteq D \cap \mathbb{C}$ we have $\operatorname{cap}\{z \in V \mid |(f - f_n)(z)| > \varepsilon\} \to 0$ as $n \to \infty$.

It follows from Definition 1.2 that (1.4) implies that $[m/n]$ converges to f in capacity in the domain $\overline{\mathbb{C}}\backslash E$. Moreover, (1.4) shows that in the Nuttall–Pommerenke Theorem convergence in capacity is faster than geometric.

The Nuttall–Pommerenke Theorem as it is stated here has been proved in [18], and it is an improvement of an earlier result by J. Nuttall [13]. It has further been extended to the class R_0 of fast approximable functions in [3]. In some respect this class R_0 has already been studied by Walsh (cf. [29, Chap. VIII]). Among other results it has been shown in [3] that fast approximable functions are necessarily single-valued. These results are in some sense the inverse of the Nuttall–Pommerenke Theorem, where single-valuedness of f in $\overline{\mathbb{C}}\backslash E$ has been assumed and fast rational approximability follows as a conclusion. Convergence results related to the class R_0 are now rather well understood.

The main interest in the present paper is the convergence of Padé approximants to functions f with branch points. If Assumption 1.1 holds and if the function f has branch points, then, as we have seen, f cannot be single-valued in $\overline{\mathbb{C}}\backslash E$. But Padé approximants are single-valued functions, and therefore it is not possible to have convergence throughout the whole domain $\overline{\mathbb{C}}\backslash E$, not even in capacity; the convergence behavior must be different from that observed in the Nuttall–Pommerenke Theorem. In the light of Gonchar's result in [3], it is clear that a convergence speed faster than geometric also is not possible.

In the first group of results we only consider functions f that satisfy Assumption 1.1. This is a rather large class; it contains, for instance, all algebraic functions analytic at infinity. In case of an algebraic function f the set E is finite, and therefore of capacity zero. On the other hand a function like

$$f(z) = \sum_{j=1}^{\infty} z^{-2^j}, \qquad |z| > 1,$$

does not satisfy Assumption 1.1 since because of the gaps in the power series the unit circle $\partial\mathbb{D}$ is the natural boundary for analytic continuations of f, and $\operatorname{cap}(\partial\mathbb{D}) = 1 > 0$.

THEOREM 1.1. *Let f satisfy Assumption* 1.1. *Then there exists a domain $D = D_f \subseteq \overline{\mathbb{C}}$, $\infty \in D$, which is unique up to a set of capacity zero, and*

(i) *the sequence $\{[m_j/n_j]\}_{j \in \mathbb{N}}$ of Padé approximants converges in capacity to f in the domain D for any sequence $\{(m_j, n_j)\}_{j \in \mathbb{N}}$ of indices satisfying*

$$m_j + n_j \to \infty, \qquad \frac{m_j}{n_j} \to 1 \qquad \textit{as} \quad j \to \infty. \tag{1.6}$$

(ii) *If $\tilde{D}$ is a domain with $\tilde{D} \supseteq D$ and $\operatorname{cap}(\tilde{D} \setminus D) > 0$, then no sequence of Padé approximants $\{[m_j/n_j]\}_{j \in \mathbb{N}}$ satisfying* (1.6) *converges in capacity to f in the whole domain $\tilde{D}$.*

Remarks. (1) Despite the assumption that f is analytic at infinity, it is not excluded that $\infty \in E$. However, if f is single-valued in $\mathbb{C} \setminus E$, then we always have $\infty \notin E$.

(2) If the function f is single-valued in $\overline{\mathbb{C}} \setminus E$, then Theorem 1.1 is a special case of the Nuttall–Pommerenke Theorem, and $D_f = \overline{\mathbb{C}} \setminus E$. Note that in the Nuttall–Pommerenke Theorem a much larger class of sequences is admissible than in Theorem 1.1. While (1.5) allows all sectorial sequences, by (1.6) only close-to-diagonal sequences are admissible. Essentially non-diagonal sequences will not be considered in the present paper.

(3) In Theorem 1.1 the convergence domain $D = D_f$ is determined only up to a set of capacity zero. Note that if B_1 and B_2 are two Borel sets, one of which is of capacity zero, then $\operatorname{cap}(B_1 \cup B_2) = \operatorname{cap}(B_1) + \operatorname{cap}(B_2)$ (cf. [12, Chap. V, Sect. 4]). Any denomberable set is of capacity zero, while, for instance, any continuum that is not reduced to a single point, and therefore also every piece of a line, is of positive capacity.

(4) Linear and planar Lebesgue measure dominate capacity. For instance, for any closed domain $V \subseteq \mathbb{C}$ we have

$$\operatorname{cap}(V) \leqslant \frac{1}{\pi} \operatorname{Area}(V) \tag{1.7}$$

(cf. [8, Chap. II, Sect. 4]). Therefore, convergence in capacity implies convergence in planar Lebesgue measure.

If the function f has branch points, then the set $\overline{\mathbb{C}} \backslash D$ contains at least one continuum that is not reduced to a single point, which implies that $\operatorname{cap}(\partial D) > 0$. As a consequence we know that in this case the Green function $g_D(z, w)$ exists in $D = D_f$ in a proper sense. (For a definition of the Green function cf. [28, Appendix V]). In order to avoid exceptional cases in the following, we extend the definition of the Green function, and define $g_D(\cdot, w) :\equiv \infty$ for all $w \in D$ if $\operatorname{cap}(\partial D) = 0$. The function G_D is then defined as

$$G(z) = G_D(z) := \exp[-g_D(z, \infty)]. \tag{1.8}$$

The function satisfies $0 \leqslant G(z) < 1$ for all $z \in D$, and we have $G(z) > 0$ for $z \in D \backslash \{\infty\}$ if $\operatorname{cap}(\partial D) > 0$. Note that ∞ is always contained in the convergence domain D. We have $G_D(z) = 1$ for quasi every $z \in \partial D$ (cf. [28, Appendix V]). A property is said to hold *quasi everywhere* (for short, qu.e.) on a set $S \subseteq \overline{\mathbb{C}}$ if it holds for every $z \in S$ with possible exceptions on a set of (outer) capacity zero.

There is a simple representation of the function (1.8) if the domain $D \subseteq \overline{\mathbb{C}}$ is simply connected: Let $\varphi: D \to \{|w| < 1\}$ be a Riemann mapping function with $\varphi(\infty) = 0$, then $G_D = |\varphi|$. If the function f has no branch points, then $D_f = \overline{\mathbb{C}}$ and consequently $G_{D_f}(z) = 0$ for all $z \in \overline{\mathbb{C}}$. Because of the extended definition of the Green function the function G_D is defined for any domain $D \subseteq \overline{\mathbb{C}}$ with $\infty \in D$. Since $g_D(z, w) = g_{\tilde{D}}(z, w)$ for all $z, w \in D \cap \tilde{D}$ if $\operatorname{cap}(D \backslash \tilde{D}) = \operatorname{cap}(\tilde{D} \backslash D) = 0$, the function G_D does not change if the two domains D and $\tilde{D}$ differ only by a set of capacity zero.

The next theorem is a quantitative strengthening of Theorem 1.1. The rate of convergence is given together with a statement in part (ii), which shows that this rate is best possible for convergence in capacity.

THEOREM 1.2. *Let f satisfy Assumption* 1.1, *let $D = D_f$ be the convergence domain from Theorem* 1.1, *$G = G_D$ the function* (1.8), *and $\{[m_j/n_j]\}_{j=1}^{\infty}$ a close-to-diagonal sequence of Padé approximants, i.e., the sequence of indices $\{(m_j/n_j)\}$ satisfies* (1.6). *Then*

(i) *for every* $\varepsilon > 0$ *and every compact sets* $V \subseteq D \backslash \{\infty\}$ *we have*

$$\lim_{j \to \infty} \text{cap}\{z \in V \mid |(f - [m_j/n_j])(z)| > (G(z) + \varepsilon)^{m_j + n_j}\} = 0, \tag{1.9}$$

(ii) *and if the function f has branch points, i.e. if* $G \not\equiv 0$, *then for every compact set* $V \subseteq D \backslash \{\infty\}$ *and every* $0 < \varepsilon \leqslant inf_{z \in V}\, G(z)$ *we have*

$$\lim_{j \to \infty} \text{cap}\{z \in V \mid |(f - [m_j/n_j])(z)| < (G(z) - \varepsilon)^{m_j + n_j}\} = 0. \tag{1.10}$$

Remarks. (1) Since $G(z) < 1$ for all $z \in D$, limit (1.9) implies that the sequence $\{[m_j/n_j]\}$ converges in capacity to f in the domain $D = D_f$. If the function f has branch points, then $0 < G(z)$ for all $z \in D \backslash \{\infty\}$, and the two limits (1.9) and (1.10) together show that the sequence $\{[m_j/n_j]\}$ converges with a geometric speed, only that there may exist exceptional sets of asymptotically vanishing capacity. The degree of convergence is given by the function $G_D(z)$ for every $z \in D_f$.

(2) It follows from Theorems 1.3 and 1.4, below, that if the function f has no branch points, i.e. if f is single-valued in $\overline{\mathbb{C}} \backslash E$, then we have $D = D_f = \overline{\mathbb{C}}$. We further have $G_D(z) = 0$ for all $z \in D = \overline{\mathbb{C}}$, and therefore in this case limit (1.9) is identical with limit (1.4) in the Nuttall–Pommerenke Theorem. However, the Nuttall–Pommerenke Theorem has been proved for sectorial sequences $\{[m_j/n_j]\}$, while Theorem 1.2 can be proved only for close-to-diagonal sequences. A convergence in capacity faster than geometric is typical for the Nuttall–Pommerenke Theorem. A more detailed investigation of the convergence speed under the assumptions of this theorem can be found in [Ka].

(3) In Theorem 1.1 and 1.2 nothing is said about the convergence in capacity for essentially non-diagonal sequences of Padé approximants. It seems to be possible to prove results that are in some way analogous to those in Theorems 1.1 and 1.2 also for functions f with branch points, but then the convergence domain will be different from D_f. The new domain will depend on the parameter λ in (1.5), and (1.5) has to be made more precise. There will be a domain of divergence that contains a neighborhood of the origin. Note that it follows from Theorem 1.5, below, that in the case of close-to-diagonal sequences we have $\overline{D_f} = \overline{\mathbb{C}}$, and therefore a non-empty domain of divergence does not exist in the case of Assumption 1.1.

(4) One may ask whether the assumption $\text{cap}(E) = 0$ is necessary. In [9] and [19] it has been shown by counterexamples that if a function f or its analytic continuation has a set of singularities $\tilde{E} \subseteq \mathbb{C}$ of positive capacity, then for the diagonal sequence of Padé approximants $[n/n]$,

$n=1, 2, ...,$ convergence in capacity may not hold on any subdomain of $\overline{\mathbb{C}}\backslash\tilde{E}$.

(5) Despite the counterexamples given in [9] and [19] it will be shown in Theorem 1.7, below, that the results of Theorems 1.1 and 1.2 can be generalized to a larger class of functions f than those satisfying Assumption 1.1. However, then the domain D of definition of the function f has to be specified first; it has to possess special properties (the symmetry property), and the boundary values of f have to satisfy certain conditions.

(6) Another question is whether the type of convergence proved in Theorems 1.1 and 1.2 can be strengthend. This may be possible, but locally uniform convergence in D is in general not true. Let for instance the function f be defined by

$$f(z)=\int_{-1}^{1} \frac{(t-\cos(\alpha_1\pi))(t-\cos(\alpha_2\pi))}{\pi\sqrt{1-t^2}(t-z)}\,dt, \tag{1.11}$$

where the three numbers 1, α_1, α_2 $(0<\alpha_1<\alpha_2<1)$ are assumed to be rationally independent. Then the function f has analytic continuations throughout $\overline{\mathbb{C}}\backslash\{-1, 1\}$. Thus, $E=\{-1, 1\}$, and Assumption 1.1 is satisfied. In [20] it has been shown that the set of all poles of the diagonal Padé approximants $[n/n]$, $n=1, 2, ...,$ are dense in $\overline{\mathbb{C}}$. Therefore, locally uniform convergence is impossible everywhere in $\overline{\mathbb{C}}$. On the other hand, it follows rather immediately from Theorem 1.3 or 1.4, below, that $D_f=\overline{\mathbb{C}}\backslash[-1, 1]$, and hence the sequence $\{[n/n]\}$ converges in capacity to f in $\overline{\mathbb{C}}\backslash[-1, 1]$.

(7) In [11] it has been shown that if a sequence $\{f_n\}_{n=1}^{\infty}$ converges in capacity to f in a domain $D\subseteq\overline{\mathbb{C}}$, then there always exists an infinite subsequence $\{f_n\}_{n\in N}$, $N\subseteq\mathbb{N}$, which converges to f qu.e. in D. The situation is analogous to that of convergence in measure and convergence almost everywhere, where also the first convergence implies the second one for subsequences.

(8) For many purposes convergence in capacity is not good enough. However, there exists a pole-elimination procedure that allows one to derive rational functions from given Padé approximants $[m_j/n_j]$, and the new sequence of rational functions converges uniformly on given compact sets in D_f. The pole elimination can be done in such a way that the speed of convergence in the uniform norm is almost the same as that in capacity.

The next three theorems are concerned with the structure of the convergence domains. The geometric aspects of these results have been studied in [21–23]; they will be repeated here for the convenience of the reader.

THEOREM 1.3 ([21; 22, Theorems 1 and 2]. *Let the function f be analytic at infinity. Then there uniquely exists a domain* $D \subseteq \overline{\mathbb{C}}$ *satisfying the following three conditions:*

(i) $\infty \in D$ *and the function f has a single-valued analytic continuation in D.*

(ii) $\operatorname{cap}(\partial D) = \inf_{\tilde{D}} \operatorname{cap}(\partial \tilde{D})$, *where the infimum extends over all domains* $\tilde{D} \subseteq \overline{\mathbb{C}}$ *satisfying assertion* (i).

(iii) $D = \bigcup \tilde{D}$, *where the union extends over all domains* $\tilde{D} \subseteq \overline{\mathbb{C}}$ *satisfying the assertions* (i) *and* (ii).

Remark. The uniquely existing domain D in Theorem 1.3 is called the *extremal* domain for single-valued analytic continuation of the function f. Of the three conditions in Theorem 1.3 the third one is of minor importance; without it the domain D is determined only up to a set of capacity zero.

THEOREM 1.4. *If the function f satisfies Assumption* 1.1, *then the convergence domain* D_f *of Theorem* 1.1 *is identical with the extremal domain D of Theorem* 1.3 *up to a set of capacity zero.*

Remarks. (1) It has already been mentioned earlier that because of the single-valuedness of the approximants $[m_j/n_j]$ the function f has also to be single-valued in the convergence domain D_f. Therefore it is not surprising that condition (i) in Theorem 1.3 plays a role in the characterization of D_f.

(2) From the definition of the Green function (cf. [28, Appendix V]), Robin's constant $r = -\log \operatorname{cap}(\partial D_f)$, and the function $G_D(z)$ in (1.8) it follows that

$$G_D(z) = \frac{\operatorname{cap}(\partial D)}{|z|} + O(z^{-2}) \qquad \text{as} \quad z \to \infty. \tag{1.12}$$

Thus, condition (ii) in Theorem 1.3 means that the domain D is chosen in such a way that the convergence factor G_D is as small as possible near infinity. This property corresponds with the intuitive understanding of Padé approximants. These approximants try to have a contact as good as possible to the function f near infinity (the point of development) for given degrees.

Theorems 1.1, 1.2, and 1.4 will be proved together with Theorems 1.7, 1.8, and 1.9 in the last section of the paper. The presentation of results will be continued by two theorems about the structure of the extremal domains D. The knowledge of the structure is the basis for the second group of convergence results.

THEOREM 1.5 (Structure Theorem [23, Theorem 1]). *Let f satisfy Assumption* 1.1, *and let $D = D_f$ be the extremal domain from Theorem* 1.3. *Then the complement $F := \overline{\mathbb{C}} \backslash D$ has empty interior and it has the structure*

$$F = F_0 \cup \bigcup_{j \in I} J_j, \tag{1.13}$$

where $F_0 \subseteq \mathbb{C}$ is a compact set with $\mathrm{cap}(F_0) = 0$, *$F_0 \backslash E$ consists of isolated points, and the J_j, $j \in I$, are open analytic arcs. The family $\{J_j\}_{j \in I} \neq \emptyset$ if and only if the function f has branch points.*

Remarks. (1) If the function f is single-valued in $\overline{\mathbb{C}} \backslash E$, then $I = \emptyset$ and $F_0 = E$.

(2) If the function f is not single-valued in $\overline{\mathbb{C}} \backslash E$, then because of $\mathrm{cap}(F_0) = 0$ and $\mathrm{cap}(J_j) > 0$ for all $j \in I \neq \emptyset$, the dominant part of F is the piece-wise analytic arcs J_j.

In the first part of the next theorem a symmetry property of the Green function $g_D(z, \infty)$ will be formulated. This property will turn out to be fundamental for the proofs of all results in the present paper.

THEOREM 1.6 (Symmetry Property [23, Theorem 1 and Corollary]). *Let the function f, the domain $D = D_f$, and its complement F be the same as in Theorem* 1.5, *and assume that the function f has branch points.*

(a) *The Green function $g_D(z, \infty)$ possesses the symmetry property*

$$\frac{\partial}{\partial \mathrm{n}_+} g_D(z, \infty) = \frac{\partial}{\partial n_-} g_D(z, \infty) \qquad \text{for all} \quad z \in J_j,\ j \in I, \tag{1.14}$$

where $\partial/\partial n_+$ and $\partial/\partial n_-$ denote the normal derivatives from both sides of the arcs J_j.

(b) *Let $h_D^*(z, \infty)$ be the conjugate harmonic function to $g_D(z, \infty)$* (*which is not single* − *valued*) *and define*

$$Q(z) := [(g_D(z, \infty) + i h_D^*(z, \infty))']^2 \qquad \text{for} \quad z \in \overline{\mathbb{C}} \backslash \mathrm{F}_0, \tag{1.15}$$

then Q is analytic in $\overline{\mathbb{C}} \backslash F_0$ and has a zero of order 2 at infinity. Let $h_D^*(z, \infty)$ be normalized in such a way that $Q(z)/z^2|_{z=\infty} > 0$. Then the arcs J_j, $j \in I$, are trajectories the quadratic differential $Q(z)\, dz^2$, more precisely

$$Q(\alpha_j(t))\, \alpha_j'(t)^2 \leqslant 0 \qquad \text{for} \quad t \in [0, 1],\ j \in I, \tag{1.16}$$

where α_j: $[0, 1] \to \mathbb{C}$ is a smooth representation of the arc J_j.

Remarks. (1) Since the arcs J_j, $j \in I$, are analytic, the Green function $g_D(z, \infty)$ can be continued harmonically across each arc J_j from both sides by reflection. Especially, it follows that the normal derivatives $\partial/\partial n_+$ and $\partial/\partial n_-$ of $g_D(z, \infty)$ from both sides of J_j exist for all $z \in J_j$, $j \in I$.

(2) The proof of part (b) of Theorem 1.6 follows from Theorem 1 in [24]. Part (a) of Theorem 1.6 follows from (1.5) in the corollary to Theorem 6 in [24].

(3) If the function f has branch points, then $I \neq \emptyset$ and $\operatorname{cap}(\partial D) > 0$. It is immediate that all points $z \in J_j$, $j \in I$, are regular with respect to the solution of Dirichlet problems in D. If there exist irregular points, then they have to belong to F_0. As a consequence we have $g_D(z, \infty) = 0$ for all $z \in J_j$, $j \in I$, and because of (1.8) $G_D(z) = 1$ for all $z \in J_j$, $j \in I$. Since $g_D(z, \infty)$ can be harmonically continued across the arcs J_j, $j \in I$, it follows from the chain rule that the symmetry property (1.14) is equivalent to

$$\frac{\partial}{\partial n_+} G_D(z) = \frac{\partial}{\partial n_-} G_D(z) \qquad \text{for all} \quad z \in J_j, \ j \in I, \tag{1.17}$$

whereas in (1.14) $\partial/\partial n_+$ and $\partial/\partial n_-$ denote the normal derivatives from both sides of J_j.

(4) If the function f has no branch points, then $I = \emptyset$, and therefore the assertions (1.14) and (1.15) are empty. Thus, in a formal sense, Theorem 1.6 holds also if f has no branch points.

(5) By a variational argument it can be shown that the symmetry property (1.14) (or (1.17)) is equivalent to the minimality of $\operatorname{cap}(\partial D)$, i.e. it is equivalent to condition (ii) in Theorem 1.3.

(6) The quadratic differential (1.16) is especially simple if f is an algebraic function since then E is finite and the function Q defined in (1.15) is rational.

Before we come to the second group of convergence results, we consider two functions f_1 and f_2 and their associated domains of convergence as examples. In order to have simple descriptions for the analytic arcs that constitute the complement of the convergence domains D_{f_1} and D_{f_2}, functions are chosen with many symmetries.

EXAMPLE 1.1. Let f_1 be defined by

$$f_1(z) := \sqrt{1 - \frac{2}{z^2} + \frac{9}{z^4}}. \tag{1.18}$$

The function has the 4 branch points $z_{1,\dots,4} = \pm\exp(\pm i\pi/6)$, and $E = \{z_1, \dots, z_4, 0\}$. Let $D_1 = D_{f_1}$ denote the extremal domain for analytic continuation of f_1. Then $D_1 := \overline{\mathbb{C}}\backslash(C_{11} \cup C_{12} \cup \{0\})$ is essentially doubly connected, and the two components C_{11} and C_{12} are trajectories of a quadratic differential. We then have

$$\frac{z^2}{z^4 - 2z^2 + 9}\, dz^2 \leqslant 0, \tag{1.19}$$

where dz is the line element on C_{11} and C_{12}. The arc C_{11} connects $z_1 = e^{i\pi/6}$ with $z_4 = e^{-i\pi/6}$ and C_{12} connects $z_2 = -e^{i\pi/6}$ with $z_3 = -e^{-i\pi/6}$. The function G_D is given by

$$G_D(z) = \exp\left[-\operatorname{Re}\int_{z_1}^{z} \frac{\zeta\, d\zeta}{\sqrt{9 - 2\zeta^2 + \zeta^4}}\right], \tag{1.20}$$

where the integral is taken along a path in $\overline{\mathbb{C}}\backslash(C_{11} \cup C_{12} \cup \{0\})$. The trajectories C_{11} and C_{12} are shown in Fig. 1 together with the poles of the Padé approximant $[40, 40]$ to the function f_1.

It is a consequence of Theorems 1.1, 1.2, and 1.4 that the diagonal Padé approximants $[n/n]$, $n = 1, 2, \dots$, converge in capacity to f_1 in the domain $D_1 = \overline{\mathbb{C}}\backslash(C_{11} \cup C_{12} \cup \{0\})$. In the special case of the function f_1 more can be said about the convergence of $\{[n/n]\}$ to f_1. It follows from the analysis contained in [2] and the symmetries of the function f_1 that we have locally uniform convergence in the domain $\overline{\mathbb{C}}\backslash(C_{11} \cup C_{12} \cup \{0\})$. This, however, is

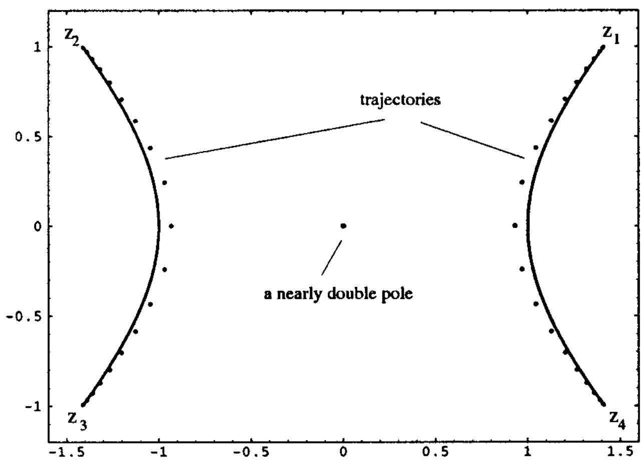

FIG. 1. Set of minimal capacity associated with function f_1 and the poles of the Padé approximant [40/40].

a result which can only be proved by using special properties of elliptic functions.

EXAMPLE 1.2. Let f_2 be defined by

$$f_2(z) := \sqrt[4]{1 - \frac{2}{z^2} + \frac{9}{z^4}}. \tag{1.21}$$

Like f_1, f_2 also has the four branch points $z_{1, ..., 4} = \pm\exp(\pm i\pi/6)$, and again $E = \{z_1, ..., z_4, 0\}$. However, the extremal domain D_2 for analytic continuation of f_2 has now to be simply connected (except for the pole at the origin). Therefore the complement of D_2 is a continuum that connects all four branch points $z_1, ..., z_4$. Finding a continuum of minimal capacity that connects a given finite set of points in $\mathbb{C}$ is known in geometric function theory as Lavrentiev's problem (cf. [5, Chap. IV]). In Fig. 2 the continuum is shown, it consists of five arcs, which are all trajectories of the same rational quadratic differential and satisfy

$$\frac{z^2 - x^2}{z^4 - 2z^2 + 9}\, dz^2 \leqslant 0, \tag{1.22}$$

where $x \in (0, 1)$ is a fixed point. At x and $-x$ the continuum bifurcates. Assumption 1.1 is satisfied by f_2. Hence, we know that the diagonal Padé approximants $[n/n]$, $n = 1, 2, ...$ converge in capacity to f_2 in the domain D_2.

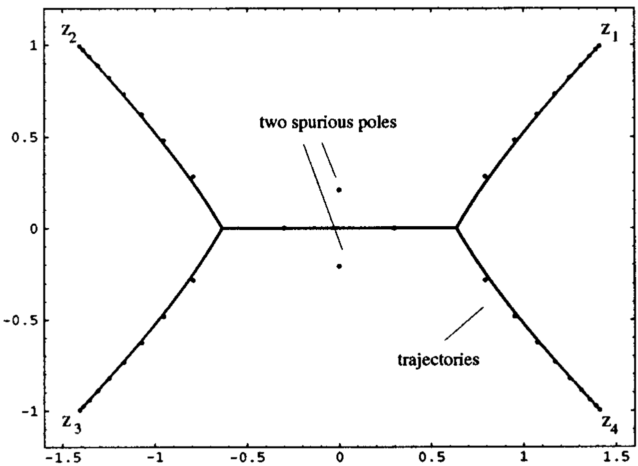

FIG. 2. Set of minimal capacity associated with function f_2 and the poles of the Padé approximant [40/40].

The function f_2 is algebraic. In [15] it has been conjectured that therefore only a bounded number of poles of each $[n/n]$ can cluster in D_2. In Fig. 2 we have plotted the poles of $[40/40]$ together with the trajectories forming ∂D_2. An inspection of the Padé approximants $[n/n]$ for other values of $n \in \mathbb{N}$ backs Nuttall's conjecture.

Next follows the second group of convergence results. For these results it is typical that the domain, in which the function f will be approximated, is specified first, and only after this are the properties of the function f specified.

DEFINITION 1.3. A domain $D \subseteq \overline{\mathbb{C}}$ is said to be *symmetric* (or to possess the symmetry property):

(i) if $\infty \in D$ and the complement $F := \overline{\mathbb{C}} \backslash D$ is of form (1.13), i.e.,

$$F = F_0 \cup \bigcup_{j \in I} J_j, \tag{1.24}$$

where $F_0 \subseteq \mathbb{C}$ is a compact set with $\operatorname{cap}(F_0) = 0$, the J_j, $j \in I$, are open analytic arcs, and $\bigcup_{j \in I} J_j \neq \emptyset$,

(ii) further, if the Green function $g_D(z, \infty)$ possesses the symmetry property (1.14).

Remark. In Theorems 1.4 through 1.6 it has been shown that if the function f satisfies Assumption 1.1 and if f has branch points, then the convergence domain D_f is symmetric.

The next theorem is the main convergence result in the second group and together with Theorem 1.2 it represents the main results of the present paper.

THEOREM 1.7. *Let $D \subseteq \overline{\mathbb{C}}$ be a symmetric domain and assume that*

(i) *f is a function analytic and single-valued in D,*

(ii) *there exists a compact set $F_1 \subseteq F$ with* $\operatorname{cap}(F_1) = 0$ *such that f has a continuous extension to every $z \in \partial D \backslash F_1$, and*

(iii) *the jump functions g_j, $j \in I$, which are defined for every $z \in J_j \backslash F_1$ by*

$$g_j(z) := f_+(z) - f_-(z), \qquad z \in J_j,\ j \in I, \tag{1.25}$$

are assumed to be continuous and different from zero on $J_j \backslash F_1$, $j \in I$.

Then any sequence $\{[m_j/n_j]\}$ of Padé approximants satisfying (1.6) *converges in capacity to f in D. Moreover, the limits* (1.9) *and* (1.10) *hold true.*

In (1.25) *the boundary values of* f *from both sides of the arc* J_j, $j \in I$, *are denoted by* f_+ *and* f_-.

Remarks. (1) We have seen in Theorems 1.3 through 1.6 that if the function f satisfies Assumption 1.1 and has branch points, then the convergence domain D_f is symmetric, and it follows from the analyticity of f in $\overline{\mathbb{C}} \backslash E$ demanded in Assumption 1.1 that the jump functions g_j, $j \in I$, defined by (1.25) are analytic on J_j and are not identical to zero. (The last conclusion is a consequence of the minimality of $\text{cap}(\partial D)$.) Thus, any function f satisfying Assumption 1.1 and having branch points fulfills also the three assumptions of Theorem 1.7.

(2) It is immediate that Theorem 1.7 covers a much larger class of functions f than do Theorems 1.1 and 1.2. Moreover, Theorem 1.7 in combination with Theorems 1.3 through 1.6 shows that the symmetric domain of Definition 1.3 can be considered to be characteristic for covergence domains of close-to-diagonal Padé approximants.

(3) The assumptions made in condition (iii) of Theorem 1.7 can certainly be weakened, but some type of condition is necessary; this follows from a class of functions studied in [24]. There it has been proved that there exist complex-valued measures μ on $[-1, 1]$ such that diagonal Padé approximants $[n/n]$, $n = 1, 2, ...,$ to the function

$$f(z) = \int \frac{d\mu(\mathrm{t})}{t - z} \tag{1.26}$$

do not converge in capacity in any subdomain of $\overline{\mathbb{C}} \backslash [-1, 1]$. The domain $\overline{\mathbb{C}} \backslash [-1, 1]$ possesses the symmetry property.

(4) Any domain of the form $\overline{\mathbb{C}} \backslash (I_1 \cup \cdots \cup I_m)$, where $I_1, ..., I_m$ are compact real intervals possesses the symmetry property. It follows from Markov's Theorem (cf. [28, Chap. 6.1]) that if the functions g_j, $j = 1, ..., m$, in (1.25) are purely imaginary and of identical sign on all intervals $I_1, ..., I_m$, close-to-diagonal sequences of Padé approximants $\{[m/n]\}$ converge not only in capacity, but also converge locally uniformly in $\overline{\mathbb{C}} \backslash \text{Co}(I_1 \cup \cdots \cup I_m)$, where $\text{Co}(I_1 \cdots I_m)$ is the convex hull of $I_1 \cdots I_m$. A result proved in [10] shows that Markov's Theorem (i.e. locally uniform convergence of diagonal Padé approximants to functions of type (1.26) with a positive measure μ) holds also in case of functions (1.26) with a complex-valued measure μ on $[-1, 1]$ if the measure μ has a continuous argument and satisfies some other conditions.

(5) If the domain $D \subseteq \overline{\mathbb{C}}$ is of the form $\overline{\mathbb{C}} \backslash E$ with $E \subseteq \mathbb{C}$ compact and $\text{cap}(E) = 0$, then D is not a symmetric domain in the literal sense of Definition 1.3. However, if f is analytic and single-valued in $\overline{\mathbb{C}} \backslash E$, then the

conclusion of Theorem 1.7 follows as a special case from the Nuttall–Pommerenke Theorem.

The last results of the present chapter are concerned with the asymptotic distribution of poles and zeros of Padé approximants $[m_j/n_j]$ and some related questions. It is clear that the asymptotic distribution of poles and zeros is decisive for the convergence of the Padé approximants.

Let $\mathscr{P}([m_j/n_j])$ and $Z([m_j/n_j])$ denote the sets of poles and zeros, respectively, of $[m_j/n_j]$, taking account of multiplicities, i.e., $\mathscr{P}([m_j/n_j])$ and $Z([m_j/n_j])$ are in general multisets. Further, let $\pi_{m_j n_j}$ and $\zeta_{m_j n_j}$ denote the measures that have a mass at each pole and zero equal to its order, i.e.,

$$\pi_{m_j n_j} := \sum_{t \in \mathscr{P}([m_j/n_j])} \delta_t, \qquad \zeta_{m_j n_j} := \sum_{t \in Z([m_j/n_j])} \delta_t. \tag{1.27}$$

The equilibrium distribution on the compact set F is denoted by $\omega = \omega_F$. (For a definition see [28, Appendix IV].) A sequence of measures μ_n, $n \in \mathbb{N}$, is said to converge weakly to a measure μ, written $\mu_n \xrightarrow{*} \mu$, if for any function h continuous on $\overline{\mathbb{C}}$ we have $\int h\, d\mu \to \int h\, d\mu$ as $n \to \infty$.

THEOREM 1.8. *Let the function f satisfy the assumptions of Theorem* 1.1 *or the assumptions of Theorem* 1.7, *and assume further that f has branch points in the case of the assumptions of Theorem* 1.1. *Let F denote the complement of D_f, where D_f is either the convergence domain of Theorem* 1.1 *or the symmetric domain in Theorem* 1.7. *Then we have*

$$\frac{1}{n_j}\, \pi_{m_j n_j} \xrightarrow{*} \omega_F, \; \frac{1}{n_j}\, \zeta_{m_j n_j} \xrightarrow{*} \omega_F \qquad \text{as} \quad m_j + n_j \to \infty. \tag{1.28}$$

Remark. The limits in (1.28) show that almost all poles and zeros of the Padé approximants $[m_j/n_j]$ cluster on the set $F = \mathbb{C} \backslash D_f$. There, they are distributed asymptotically like the equilibrium distribution ω_F. In [15] a more specific conjecture has been made. It is conjectured that if f is an algebraic function, then only a bounded number of poles of each approximant $[m_j/n_j]$ can cluster inside of D.

Practically as a byproduct of the last theorem we can deduce information about the asymptotic ($j \to \infty$) degree of the approximants $[m_j/n_j]$, and similarly informations about the contact of the approximants with the function f at infinity. It has been mentioned already at the beginning of this paper that the Padé approximant $[m_j/n_j]$ may have a contact with f at infinity that is less than $m_j + n_j + 1$, which one would expect in accordance with the number of free coefficients in $[m_j/n_j]$. Of course, the contact can also be greater. By $d_{m_j n_j} \in \mathbb{Z}$ we denote the interpolation defect, i.e.,

$m_j + n_j + 1 - d_{m_j n_j}$ is the order of the zero that the error function $f - [m_j/n_j]$ has at infinity.

THEOREM 1.9. *Under the same assumptions as in Theorem* 1.8 *we have*

$$\lim_{j\to\infty} \frac{1}{n_j}\, d_{m_j n_j} = 0 \tag{1.29}$$

and

$$\lim_{j\to\infty} \frac{1}{n_j}\, \deg([m_j/n_j]) = 1. \tag{1.30}$$

After the first submission of the present paper to the *Journal of Approximation Theory*, papers [25–27] and [6] have appeared with results related to and also generalizing the theorems stated so far. While in [25–27] only the convergence of multi-point Padé approximants associated with best rational approximants has been studied, the paper [6] contains a rather comprehensive investigation covering several directions of the topic. The main subject, perhaps, is the solution of the famous "1/9"-problem, which involves the use of multi-point Padé approximants. In Section 3 of that paper results were proved which overlap the material given here. However, the results presented there are in a more implicit and compressed form, and the proofs follow different paths, which may be shorter, but since the subject is technically rather difficult it may be helpful to have a different treatment, which presents at the same time our original approach. While our approach has not changed, the organization of the proofs in the present version of the paper differs from that originally submitted. It is hoped that the new version offers an easier and more accessible presentation of the material.

The study of Padé approximants to functions with branch points was pioneered by J. Nuttall in [16], and the connection to sets of minimal capacity was shown and explored for the first time in [Nu2]. For a survey we recommend [4].

The outline of the paper is as follows: In the next section we introduce and discuss some tools basic to everything that follows. Especially, we assemble results from potential theory, which differ somewhat from those given in the usual form. Because of a special normalization used for polynomials, logarithmic potentials also will be defined in a non-standard way, which makes it necessary to review the central results from potential theory and adapt them to the new situation. Besides that, results are proved that are connected with the symmetry property (1.14). In Section 3 we prove four auxiliary lemmas that are fundamental for the proofs in the last section. Most important are the first two lemmas about the existence of two special potentials. The proofs of the main results are given in Section 4.

2. NOTATIONS AND SOME RESULTS FROM POTENTIAL THEORY

Notation about and tools for normalization of polynomials, logarithmic potentials, the structure of symmetric domains D and their complementary sets F are introduced. A special normalization of polynomials and potentials is necessary since in our investigations the zeros of polynomials and correspondingly the masses of potentials may tend to infinity. Classical results from potential theory are reviewed and presented in a way that takes care of the changes caused by the special normalization. A major topic is also the definition and discussion of a so called Φ-symmetry and its immediate consequence, which is connected to the symmetry property of domains.

Denote by $\mathbb{D}(x, r)$ the open disc around $x \in \mathbb{C}$ with radius $r > 0$, and $\mathbb{D} := \mathbb{D}(0, 1)$. The normalization of polynomials will be based on the *linear factor* $H(z, x)$, which is defined as

$$H(z, x) := \begin{cases} z - x & \text{if} \quad x \in \bar{D} \\ (z-x)/|x| & \text{if} \quad x \in \mathbb{C} \setminus \mathbb{D} \\ 1 & \text{if} \quad x = \infty \end{cases} \tag{2.1}$$

for all $z \in \overline{\mathbb{C}}$. By $\mathscr{P}_n^* \subseteq \mathscr{P}_n$ we denote the set of all polynomials $p \in \mathscr{P}_n$ normalized in such a way that

$$p(z) = \prod_{x \in Z(p)} H(z, x), \tag{2.2}$$

where as before $Z(p)$ denotes the set of all zeros of p. The main advantage of this normalization becomes apparent if in a sequence of polynomials $p_1, p_2, ... \in \mathscr{P}_n^*$, n fixed, some zeros tend to infinity, then the polynomials p_j remain bounded on compact sets of $\mathbb{C}$ as $j \to \infty$, which is not the case, for instance, if we had chosen the polynomials $\mathscr{P}_j$ to be monic. Note that $0 \notin \mathscr{P}_n^*$.

For a measure μ supported in $\overline{\mathbb{C}}$ we define the *logarithmic* potential as

$$p(\mu; z) := \int \log \frac{1}{|H(z, x)|} \, d\mu(x). \tag{2.3}$$

This definition corresponds to the normalization of polynomials in $\mathcal{P}_n^*$. For measures μ in $\mathbb{C}$ with $\int_{\mathbb{C}\setminus\mathbb{D}} \log|x|\, d\mu(x) < \infty$ we have

$$p(\mu; z) = \int \log \frac{1}{|z-x|}\, d\mu(\mathrm{x}) + \int_{\mathbb{C}\setminus\mathbb{D}} \log|\mathrm{x}| d\mu(\mathrm{x}). \tag{2.4}$$

Usually, the logarithmic potential of a measure μ is defined as $\int \log|z-x|^{-1}\, d\mu(x)$ (*cf.*[8; 28, Appendix]), which differs from $p(\mu; z)$ by the constant $\int_{\mathbb{C}\setminus\mathbb{D}} \log|x|\, d\mu(x)$, as (2.4) shows. Of course, this constant can be infinite. The two definitions are identical if $[\mathrm{supp}](\mu) \subseteq \bar{\mathbb{D}}$. The potential $p(\mu; z)$ defined in (2.3) is never identical to infinity, even if the measure μ has a strong growth near infinity. Actually, the potential also exists if $\mu(\{\infty\}) \neq 0$. Let the measure μ be written as $\mu = \mu_1 + \mu_2$ with $\mu_1 := \mu|_{\bar{\mathbb{D}}}$ and $\mu_2 := \mu|\bar{\mathbb{C}}\setminus_{\bar{\mathbb{C}}\setminus\bar{\mathbb{D}}}$. Denote the image measure of μ_2 under the map $z \mapsto 1/z$ by $\tilde{\mu}_2$. Then we have

$$p(\mu; z) = p(\mu_1; z) + p\left(\tilde{\mu}_2; \frac{1}{z}\right) + \mu_2(\bar{\mathbb{C}}\setminus\mathbb{D}) \log\frac{1}{|z|}. \tag{2.5}$$

Note that on the right-hand side of (2.5) both measures have their support in $\bar{\mathbb{D}}$. Formula (2.5) is an immediate consequence of the identity

$$\begin{aligned} \int \log \frac{|x|}{|z-x|}\, d\mu_2(x) &= \int \log \frac{|1/\mathrm{x}|}{|z-1/x|}\, d\tilde{\mu}_2(x) \\ &= \int \log \frac{|1/z|}{|1/z-x|}\, d\tilde{\mu}_2(\mathrm{x}). \end{aligned} \tag{2.6}$$

It turns out that the classical tools of potential theory hold without change or with only minor changes for the version of the logarithmic potential introduced in (2.3). In the next lemmas we assemble results from potential theory that are needed in the subsequent analysis. The proofs will be kept as short as possible. In most cases the proof is an immediate consequence of identity (2.6). The assembling of results from potential theory demands some space, however; the modified definition (2.3) of the logarithmic potentials has some surprising consequences that have to be clarified.

All measures are assumed to be non-negative Borel measures on $\bar{\mathbb{C}}$. As in Theorem 1.8, $\xrightarrow{*}$ denotes the convergence of measures in the weak topology.

LEMMA 2.1 (Principle of Descent). *If* $\mu_n \xrightarrow{*} \mu_0$ *and* $z_n \to z_0$ *as* $n \to \infty$, z_n, $z_0 \in \mathbb{C}$, *then*

$$\liminf_{n \to \infty} p(\mu_n; z_n) \geqslant p(\mu_0; z_0). \tag{2.7}$$

Proof. The relation (2.7) is an immediate consequence of (2.5) and the principle of descent in the theory of logarithmic potentials (cf. [8, Theorem 1.3]). Note that on the right hand side of (2.5) all potentials have defining measures with supports in $\overline{\mathbb{D}}$, therefore there is no difference between thedefinition (2.3) and the classical definition of logarithmic potentials. Further, it follows from $\mu_n \xrightarrow{*} \mu_0$ that $|\mu_n| \to |\mu_0|$ as $n \to \infty$. ▮

LEMMA 2.2 (Lower Envelope Theorem). *If* $\mu_n \xrightarrow{*} \mu_0$ *as* $n \to \infty$, *then*

$$\liminf_{n \to \infty} p(\mu_n; z) = p(\mu_0; z) \tag{2.8}$$

for quasi every $z \in \overline{\mathbb{C}}$.

Proof. As in the principle of descent the lower envelope theorem follows from identity (2.5) together with the same theorem in the general theory of potentials (cf. [8, Theorem 3.8]). ▮

Remarks. (1) It is quite remarkable that the Lower Envelope Theorem given here holds without any additional condition on the measures μ_n. In the case of the classical definition it is often demanded that all measures μ_n have their supports in a common compact set $V \subseteq \mathbb{C}$, or, as in [8, Theorem 3.8], that there is a growth condition which has to be satisfied by all measures μ_n. (In [8, Theorem 3.8] only Riesz potentials have been considered; for logarithmic potentials the growth condition has to be modified appropriately.)

(2) It follows from [8, Remark 2 to Theorem 3.8] that if $u(z)$ denotes the left-hand side of (2.8) and if the so-called *lower semicontinous regularization* is defined as

$$\tilde{u}(z) := \liminf_{w \to z} u(w), \tag{2.9}$$

then $\tilde{u}(z) = p(\mu_0; z)$ for all $z \in \overline{\mathbb{C}}$.

LEMMA 2.3 (Balayage). *Let* $G \subseteq \overline{\mathbb{C}}$ *be a domain with* $\operatorname{cap}(\overline{\mathbb{C}} \backslash G) > 0$ *and* μ *a positive measure on* $\overline{\mathbb{C}}$. *Then there exists a positive measure* $\hat{\mu}$, *called the balayage measure of* μ, *with* $\operatorname{supp}(\hat{\mu}) \subseteq \overline{\mathbb{C}} \backslash G$, $\|\mu\| = \|\hat{\mu}\|$,*and* $c \in \mathbb{R}$ *such that*

$$p(\mu; z) = p(\hat{\mu}; z) + c \tag{2.10}$$

for all $z \in \overline{\mathbb{C}} \backslash \overline{G}$ *and for all regular points* $z \in \partial G$. *If* $G \subseteq \mathbb{D}L$ *or if* $G \subseteq \mathbb{C} \backslash \overline{\mathbb{D}}$ *and* ∞ *is not an irregular point of* ∂G, *then* $c = 0$. *The measure* $\hat{\mu}$ *is given by*

$$\hat{\mu} = \int \omega_x \, d\mu(x), \tag{2.11}$$

where ω_x *is the harmonic measure on* ∂G *for* $x \in (G \cup I_G)$, *and* I_G *denotes the set of all irregular points of* ∂G. *For all other points* x *we have* $\omega_x = \delta_x$ *in* (2.11). *The constant* c *is given by*

$$c = \begin{cases} \int_{\mathbb{C} \backslash \overline{\mathbb{D}}} \log |x| \, d(\mu - \hat{\mu})(\mathrm{x}) & \text{if} \quad \infty \notin G \cup I_G \\ \int_{\mathbb{C} \backslash \overline{\mathbb{D}}} \log |x| \, d(\mu - \hat{\mu})(\mathrm{x}) - \int g_G(x, \infty) \, d\mu(x) & \text{if} \quad \infty \in G \cup I_G . \end{cases} \tag{2.12}$$

Remarks. (1) In the lemma the term regular or irregular means regular or irregular, respectively, with respect to the Dirichlet problem in G (cf. [8, Chap. V]).

(2) In general, at irregular points in ∂G we do not have equality in (2.10), but rather the inequality

$$p(\mu; z) \geqslant p(\hat{\mu}; z) + c \tag{2.13}$$

for all $z \in \overline{G}$. Actually, it can be shown that in (2.13) a proper inequality holds for all irregular points z in ∂G and for all $z \in G$ if and only if $\operatorname{supp}(\mu) \cap (G \cup I_G) \neq \emptyset$.

(3) For a definition of the harmonic measure ω_x see [7, Chap. VII]. In [8, Chap. IV, Sect. 1] it is called the Green measure.

Proof. By identity (2.5) the balayage problem is carried over to a balayage problem of classical logarithmic potentials. We will discuss the procedure in some detail. Let μ_1 and $\tilde{\mu}_2$ be the two measures defined as in (2.5), and let $\tilde{G}$ be the image of G under the mapping $z \mapsto 1/z$. Let μ_1 and $\tilde{\mu}_2$ be the balayage measures resulting from sweeping μ_1 out of G and $\tilde{\mu}_2$ out of $\tilde{G}$, respectively. For a definition of balayage see [8, Chap. IV, Sect. 2]. We have

$$p(\mu_1; z) = \int \log \frac{1}{|z - x|} \, d\hat{\mu}_1 \, (\mathrm{x}) + \mathrm{c}_1 \tag{2.14}$$

for all $z \in \overline{\mathbb{C}} \backslash \overline{G}$ and all regular points of ∂G, and

$$p(\tilde{\mu}_2; z) = \int \log \frac{1}{|z - x|} \, d\hat{\tilde{\mu}}_2(\mathrm{x}) + \mathrm{c}_2 \tag{2.15}$$

for all $z \in \overline{\mathbb{C}} \setminus \overline{\widetilde{G}}$ and all regular points of $\partial \widetilde{G}$. Since all objects related with harmonic functions, like harmonic measures, regular and irregular points, etc., are carried over by the mapping $z \mapsto 1/z$, identity (2.10) follows from (2.5), (2.14), and (2.15) if we denote the image of $\hat{\tilde{\mu}}_2$ under $z \mapsto 1/z$ by $\hat{\mu}_2$ and set $\hat{\mu} = \hat{\mu}_1 + \hat{\mu}_2$. The formula for the constant c given in (2.12) can be proved by using the formula for the balayage in the case of the classical definition (cf. [8, formula (4.2.6)]) and then making the changes resulting from the definition (2.3). ∎

The next lemma is given here not in its most general form, but it is sufficient for our needs.

Lemma 2.4 (Principle of Domination). *Let ν, μ be measures with $\|\nu\| \geqslant \|\mu\|$ and $c \in \mathbb{R}$ a constant. If the measure ν is of finite energy,* $\operatorname{supp}(\nu) \subseteq \mathbb{C}$ *compact, and*

$$p(\nu; z) \leqslant p(\mu; z) + c \qquad \text{for quasi every } z \in \operatorname{supp}(\nu), \tag{2.16}$$

then

$$p(\nu; z) \leqslant p(\mu; z) + c \qquad \text{for all} \quad z \in \overline{\mathbb{C}}. \tag{2.17}$$

Proof. First we suppose that both measures ν and μ have compact supports. There exist $r < 1$ such that the mapping $\psi : z \mapsto rz$ maps $\operatorname{supp}(\nu)$ and $\operatorname{supp}(\mu)$ into $\mathbb{D}$. We have

$$p(\nu; z) \leqslant p((\psi(\nu); \psi(z)) + \int_{|x| \leqslant 1/r} \log(r \max(1, |x|)) \, d\nu(x) \tag{2.18}$$

for all $z \in \mathbb{C}$. The same identity holds if ν is substituted by μ. Of course, the image measure $\psi(\nu)$ is also of finite energy. Since $\operatorname{supp}(\psi(\nu))$, $\operatorname{supp}(\psi(\mu)) \subseteq \mathbb{D}$, the logarithmic potentials defined by (2.3) and the classical definition coincide, and therefore it follows (cf. [8, Theorem 1.27]) that (2.16) implies (2.17) if we consider the potentials of the measures $\psi(\nu)$ and $\psi(\mu)$ instead of ν and μ, respectively. Together with (2.18) the conclusion then also follows for the original measures ν and μ.

If $\operatorname{supp}(\mu)$ is not compact in $\mathbb{C}$, then we consider the restricted measure $\mu_R := \mu|_{\overline{\mathbb{D}(0, R)}}$, $R > 0$. We have

$$\begin{aligned} |p(\mu; z) - p(\mu_R; z)| &\leqslant \mu(\overline{\mathbb{C}} \setminus \mathbb{D}(0, R)) \log \frac{1}{|1 - R_1/R|} \\ &\leqslant 2\mu(\overline{\mathbb{C}} \setminus \mathbb{D}(0, R)) \frac{R_1}{R} \end{aligned} \tag{2.19}$$

for $|z| \leqslant R_1 < R$, R sufficiently large and fixed. Since the right-hand side of (2.19) can be made arbitrarily small by choosing R sufficiently large, the lemma is also proved for measures μ with an unbounded support. ▮

The next lemma is a consequence of the continuity of logarithmic potentials in the fine topology (cf. [8, Chap. V, Sect. 3]). Let $f (f \not\equiv \infty)$ be superharmonic in a simply connected domain $G \subseteq \mathbb{C}$. Because of the lower semicontinuity of f the sets

$$O := O(c) := \{z \in G \mid f(z) > c\}, \qquad c \in \mathbb{R}, \tag{2.20}$$

are open in the Euclidian topology, and it follows from the superharmonicity of f that each component of O is simply connected.

LEMMA 2.5. *Assume that the function f is superharmonic and possesses a finite harmonic minorant, and let $\{O_j\}$ be the set of all components of the open set O defined by* (2.20). *If for some $x \in G$ we have $f(x) < c$, then $x \notin \partial O_j$ for any component O_j.*

Remark. Note that it cannot be excluded that $x \in \overline{O} \setminus \bigcup_j \overline{O_j}$.

Proof. Each component O_j is a domain. Hence, O_j is not thin near x for any $x \in \partial \overline{O_j}$, which follows immediately from Wiener's regularity criterion (cf. [8, Chap. V, Sect. 1 and Sect. 3]). Since f can be represented in G as the sum of a logarithmic potential and a harmonic function (cf. [8, Theorem 1.24′]), f is continuous in the fine topology (cf. [8, Chap. V, Sect. 3]). For the validity of Theorem 1.24′ in [8] it is demanded that f has a finite harmonic minorant in G. Now let now $x \in \partial O_j$. Since O_j is not thin near x it follows that x belongs to the fine closure of O_j, and the fine continuity of f together with the definition of O in (2.20) implies that $f(x) \geqslant c$, which proves the lemma. ▮

LEMMA 2.6. *Let $G \subseteq \overline{\mathbb{C}}$ be a domain with* $\operatorname{cap}(\overline{\mathbb{C}} \setminus G) > 0$. *We have*

$$g_G(z, x) = p(\delta_x - \omega_x; z) + c_x \qquad \textit{for} \quad z, x \in G, \tag{2.21}$$

where ω_x is the harmonic measure on ∂G representing the point $x \in \overline{G}$, and $c_x \in \mathbb{R}$ is a coherent depending on x. For $x \in \partial G$ we have $\delta_x = \omega_x$ if and only if x is a regular point of ∂G.

Remarks. (1) The harmonic measure ω_x is the balayage measure of the Dirac measure δ_x resulting from sweeping δ_x out of the domain G (cf. [8, Chap. IV, Sect. 1]. In [8] the harmonic measure is called the Green measure. See also (2.11) in Lemma 2.3).

(2) In the case of the classical definition of the logarithmic potential one has two different representations for the Green function $g_G(z, x)$, depending on whether $x \neq \infty$ or $x = \infty$ (cf. [28, Appendix VII, (A.18), (A.19)]). With the special definition (2.3) representation (2.21) holds for both cases $x \neq \infty$ and $x = \infty$. The constant c_x depends continuously on $x \in G$.

(3) If $x = \infty$, then the harmonic measure ω_∞ is also known as the equilibrium distribution on $\overline{\mathbb{C}} \backslash G$. If in addition $\partial G \subseteq \overline{\mathbb{D}}$, then we have $c_\infty = \log(1/\text{cap}(\partial G))$ (cf. [28, Appendix V]).

(4) If $\text{cap}(\overline{\mathbb{C}} \backslash G) > 0$ and $\infty \notin \partial G$, then the harmonic measure ω_x is of finite energy for all $x \in G$ ([28, Appendix I]).

Proof of the Lemma. The lemma follows from [8, Chap. IV, Sects. 1 and 2]. For the case $x = \infty$ [La, Chap. IV, Sect. 1], Section 3 is especially important. See also [28, Appendix VII]. ∎

The next lemma is generally not considered as a standard tool of potential theory. However, in the proofs given in Section 3 it is used at several places and plays there an essential role.

LEMMA 2.7 (Balayage with Mass Reduction). *Let $V \subseteq \mathbb{C}$ be a compact set, $\overline{\mathbb{C}} \backslash V$ connected, and* $\text{cap}(V) > 0$.

(i) *For each $x \in \overline{\mathbb{C}} \backslash V$ there exists $a(x) \in (0, 1)$ such that the measure*

$$\omega_x - a(x)\,\omega_\infty \geqslant 0 \tag{2.22}$$

and $a(x)$ is a measurable function of $x \in \overline{\mathbb{C}} \backslash V$.

(ii) *Let μ be a positive measure on $\overline{\mathbb{C}}$ with* $\text{supp}(\mu) \backslash V \neq \emptyset$. *Then there exists $\hat{\mu} \geqslant 0$ with* $\text{supp}(\hat{\mu}) \subseteq V$, $\|\hat{\mu}\| < \|\mu\|$, *and a constant $c \in \mathbb{R}$ such that*

$$p(\mu; z) = p(\hat{\mu}; z) + c \tag{2.23}$$

for all $z \in \text{Int}(V)$ and for all regular points $z \in \partial V$ (regular with respect to the Dirichlet problem in $\overline{\mathbb{C}} \backslash V$). The balayage measure is given by

$$\hat{\mu} = \int a(x)\, \omega_x \, d\mu(x), \tag{2.24}$$

where ω_x is the harmonic measure on ∂V representing x if $x \in \overline{\mathbb{C}} \backslash V$ or if x is an irregular point of ∂V, for all other x we set $\omega_x := \delta_x$, and $a(x) := 1$ for all $x \in V$.

Remark. The important difference to Lemma 2.3 is that now $\|\hat{\mu}\| < \|\mu\|$ for the balayage measure $\hat{\mu}$.

Proof. (i) It is not difficult to verify that for each $x \in \mathbb{C} \setminus V$ we can find $1 > a(x) > 0$ such that the function

$$h(z, x) := g_{\overline{\mathbb{C}} \setminus V}(z, x) - a(x)\, g_{\overline{\mathbb{C}} \setminus V}(z, \infty) \tag{2.25}$$

is nonnegative (as a function of z) everywhere in $\mathbb{C}$ except in a small neighborhood of infinity. From Lemma 2.6 we know that

$$h(z, x) = p(\delta_x - \omega_x + a(x)\, \omega_\infty; z) + c_x - a(x)\, c_\infty. \tag{2.26}$$

Since $h(\cdot, x) \geqslant 0$ in a neighborhood of V and $h(\cdot, x) = 0$ quasi everywhere on ∂V, it follows that $h(\cdot, x)$ is subharmonic in a neighborhood of V. This proves (2.22). The function $a(x) > 0$ can be chosen to be piece-wise constant on a system of measurable sets covering $\overline{\mathbb{C}} \setminus V$. Hence, we can assume that $a(x)$ is a measurable function.

(ii) Let $g_{\overline{\mathbb{C}} \setminus V}(z, x)$ be extended in the usual way to all $z, x \in \overline{\mathbb{C}}$. Then by (2.25) also $h(z, x)$ is defined for all $z, x \in \overline{\mathbb{C}}$. Set

$$h(z) := \int h(z, x)\, d\mu(x). \tag{2.27}$$

Then the function $p(\mu; z) - h(z)$ is harmonic in $\overline{\mathbb{C}} \setminus V$ and $h(z) = 0$ for $z \in \operatorname{Int}(V)$ and for all regular points $z \in \partial V$. With the measure $\hat{\mu}$ defined by (2.24) and (2.26) it follows that $p(\mu; z) - h(z) = p(\mu; z) + c$, which proves (2.23). ▮

The next results are related to symmetric domains $D \subseteq \mathbb{C}$ introduced in Definition 1.3. The complement F of the domain D, the set F_0, and the arcs J_j, $j \in I$, are defined as in (1.24). As in Definition 1.3 we assume that F contains at least one analytic arc J_j, i.e., $I \neq \emptyset$. Since all arcs J_j, $j \in I$, are open, for each J_j there exists a simply connected domain $U_j \subseteq \mathbb{C}$ such that

(i) $U_i \cap U_j = \emptyset$ for $i, j \in I$, $i \neq j$,

(ii) $U_j \cap F = J_j$ for $j \in I$, and

(iii) there exists a conformal mapping $\varphi_j \colon U_j \to \mathbb{C}$ such that

$$|\operatorname{Im} \varphi_j(z)| = g_D(z, \infty) \qquad \text{for} \quad z \in U_j,\ j \in I, \tag{2.28}$$

$\varphi_j(J_j) \subseteq \mathbb{R}$, and $\overline{\varphi_j(U_j)} = \varphi_j(U_j)$, $j \in I$.

The existence of the mappings φ_j is an immediate consequence of the symmetry property (1.14) since (1.14) implies that $g_D(z, \infty)$ can be continued harmonically across J_j by reflection. The resulting function is the imaginary part of φ_j. The mappings φ_j are determined by (2.28) up to an additive real constant. The property $\overline{\varphi_j(U_j)} = \varphi_j(U_j)$ can always be achieved by restricting a given domain U_j.

DEFINITION 2.1. Let U be the union of all domains U_j, $j \in I$, and define $\varphi\colon U \to \mathbb{C}$ by $z \mapsto \varphi_j(z)$ if $z \in U_j$. Then the *reflection function* Φ is defined as

$$\Phi(z) := \varphi^{-1}(\overline{\varphi(z)}), \qquad z \in U. \tag{2.29}$$

The function Φ is anti-analytic, i.e. $\overline{\Phi(z)}$ is analytic in U, and each point $z \in F \backslash F_0 = \bigcup_{j \in I} J_j$ is a fixpoint of Φ. The set $U_j \backslash J_j$, $j \in I$, consists of two domains, and Φ maps each of these two domains onto its counterpart. By J_{j+} and J_{j-} we denote the two banks of J_j, $j \in I$.

DEFINITION 2.2. A subset $V \subseteq U$ or a function h defined on U is *Φ-invariant* if $\Phi(V) = V$ or if $h(z) = h(\Phi(z))$ for all $z \in U$, respectively. For $x \in F \backslash F_0$ and $\delta > 0$ sufficiently small a *Φ-disc* $L(x, \delta)$ is defined as

$$L(x, \delta) := \varphi^{-1}(\mathbb{D}(\varphi(x), \delta)). \tag{2.30}$$

The parameter δ has to be chosen so small that $\mathbb{D}(\varphi(x), \delta) \subseteq \varphi(U)$. For $\delta > 0$ small $L(x, \delta)$ *is* similar to a circle, and it becomes the more so the smaller $\delta > 0$ becomes.

LEMMA 2.8. *The Green function $g_D(z, \infty)$ and the sets $L(x, \delta)$ and $(F \cup L(x, \delta)) \backslash F_0$, $x \in F \backslash F_0$, $\delta > 0$ sufficiently small, are Φ symmetric.*

Proof. The Φ-symmetry of $g_D(z, \infty)$ follows from (2.28), and the Φ-symmetry of $L(x, \delta)$ and $(F \cup L(x, \delta)) \backslash F_0$ is an immediate consequence of (2.30). ∎

We conclude this section with some more lemmas, which are related to the Green function $g_D(z, \infty)$ and the sets F and F_0.

LEMMA 2.9. *We have*

$$g_D(z, \infty) = 0 \qquad \textit{for all} \quad z \in F \backslash F_0, \tag{2.31}$$

and the Green function has harmonic continuations across each arc J_j, $j \in I$, from both sides of J_j. These continuations are determined by (2.28).

Proof. Since $F \backslash F_0 = \bigcup_j J_j$, each point $z \in F \backslash F_0$ is regular with respect to Dirichlet problems in D. Hence, (2.31) holds (cf. [8, Chap. IV, Sect. 2]). The harmonic continuation follows immediately from (2.28) and the analyticity of Φ.

LEMMA 2.10. *For any $x \in U$ the function*

$$h_x(z) := g_D(z, x) - g_D(\Phi(z), \Phi(x)), \qquad z \in U, \tag{2.32}$$

is harmonic in U and we have

$$h_x(z) = 0 \qquad \text{for all} \quad z \in F \backslash F_0. \tag{2.33}$$

For any $x_0 \in F \backslash F_0$ we have

$$\lim_{x \to x_0} h_x(z) = 0 \qquad \text{locally uniformly for} \quad z \in U. \tag{2.34}$$

Proof. It is immediate that h_x is harmonic in U, and (2.33) follows from Lemma 2.9. From the symmetry of the Green function $g_D(z, x)$ in the two variables z and x it follows that $g_D(z, x) = g_D(x, z) \to 0$ if $x \to x_0$ for each $z \in \partial U \cap D$. Because of (2.32) this limit implies $h_x(z) \to 0$ for $z \in \partial U \cap D$ if $x \to x_0$. The point-wise convergence on $\partial U \cap D$ implies the locally uniform convergence in U. ∎

LEMMA 2.1. *Let $F_1 \subseteq \mathbb{C}$, $F_1 \neq \emptyset$, be a compact set with* $\operatorname{cap}(F_1) = 0$. *Then there exists a probability measure μ_1 with* $\operatorname{supp}(\mu_1) = F_1$ *and*

$$p(\mu_1; z) = \infty \qquad \text{for all} \quad z \in F_1. \tag{2.35}$$

Proof. The lemma has been proved in [8, Theorem 3.1]. ∎

LEMMA 2.12. *For any $x \in F \backslash F_0$ there exists a probability measure μ_2 with* $\operatorname{supp}(\mu_2) \subseteq F \backslash \{x\}$ *and*

$$p(\mu_2; x) < p(\mu_2; z) \qquad \text{for all} \quad z \in F \backslash (F_0 \cup \{x\}) \tag{2.36}$$

Proof. Let $x \in J_j$ and let $\delta > 0$ be so small that $L(x, \delta) \subseteq U$. If we define $\delta_1 := \delta(1-t)$, $\delta_2 := \delta t$, $t \in (0, 1)$, then the interval $[-\delta_1, \delta_2]$ of length δ moves from left to right if t moves through $(0, 1)$ from 0 to 1. We consider the compact sets

$$F_t := F \backslash \Phi^{-1}((\Phi_j(x) - \delta(1-t), \Phi_j(x) + \delta t)), \qquad t \in (0, 1). \tag{2.37}$$

Let ω_t be the equilibrium measure on F_t, and let $c_t \in \mathbb{R}$ be such that $p(\omega_t; z) = c_t$ for all $z \in F_t \backslash F_0$. Then $p(\hat{\omega}_t; z) < c_t$ for $z \in J_j \backslash F_t$. Since J_j is an analytic arc, for $\delta > 0$ sufficiently small $p(\omega_t; z)$ has only one minimum on the subarc $J_j \backslash F_t$ for each $t \in (0, 1)$. If t moves from 0 to 1, then the minimum of $p(\omega_t; z)$ on $J_j \backslash F_t$ moves from one side of x to the other. Hence, there exists $t \in (0, 1)$ such that $p(\omega_t; z)$ has its unique minimum on $F \backslash F_0$ at x. With $\mu_2 := \omega_t$ the lemma is proved.

LEMMA 2.13. *Let the function f be superharmonic in* $\mathbb{C}$, *harmonic in* $\mathbb{C}\backslash V$ *with* $V \subseteq \mathbb{C}$ *a compact set, and assume that it has the behavior*

$$f(z) = c_1 \log \frac{1}{|z|} + O(1) \qquad as \quad z \to \infty,\ c_1 > 0. \tag{2.38}$$

Then there exists a positive measure μ_3 *with* $\|\mu_3\| = c_1$ *and* $\operatorname{supp}(\mu_3) \subseteq V$ *such that*

$$f(z) = p(\mu_3; z) + c_2 \qquad for\ all \quad z \in \mathbb{C}. \tag{2.39}$$

Proof. The lemma follows directly from Theorem 1.22′ in [8].

By the next results we built a bridge between weak convergence of measures and convergence in capacity (cf. Definition 1.2). Let f be a function superharmonic in a domain $G \subseteq \overline{\mathbb{C}}$ with $\operatorname{cap}(\partial G) > 0$ and assume that f has a finite harmonic minorant in G. Then there exists a positive measure $\nu \geqslant 0$ defined on G and a function f_G harmonic in G such that

$$f(z) = f_G(z) + g_G(\nu; z), \qquad z \in G, \tag{2.40}$$

with $g_G(\nu; z)$ denoting the Green potential

$$g_G(\nu; z) := \int g_G(z, x)\, d\nu(x) \tag{2.41}$$

(cf. [8, Theorem 1.22′]).

LEMMA 2.14. *Given a sequence* $\{f_n\}$ *of functions superharmonic in a domain* $G \subseteq \overline{\mathbb{C}}$ *with* $\operatorname{cap}(\partial \mathrm{G}) > 0$, *let the functions* $\hat{f}_n$ *be harmonic in G and* ν_n *be positive measures on G such that*

$$f_n(z) = \hat{f}_n(z) + g_G(\nu_n; z) \qquad for \quad z \in G. \tag{2.42}$$

If

(i) $\nu_n(V) \to 0$ *as* $n \to \infty$ *for all compact sets* $V \subseteq G$,

(ii) $\|\nu_n\| \leqslant c$ *for all* $n \in \mathbb{N}$,

(iii) $\lim_{n \to \infty} \hat{f}_n(z) = f(z)$ *locally uniformly for* $z \in G$, *where f is a function harmonic in G and* $f \not\equiv -\infty$, *then* f_n *converges to f in capacity in G, i.e. for any* $\varepsilon > 0$ *and any compact set* $V \subseteq G \cap \mathbb{C}$, *we have*

$$\lim_{n \to \infty} \operatorname{cap}\{z \in V \mid |(f_n - f)(z)| > \varepsilon\} = 0. \tag{2.43}$$

COROLLARY 2.15. *If $\mu_n \overset{*}{\to} \mu$ as $n \to \infty$, then $p(\mu_n; \cdot)$ converges to $p(\mu; \cdot)$ in capacity in $\overline{\mathbb{C}} \backslash \operatorname{supp}(\mu)$.*

Proof of the Corollary. Let G be a domain with $\bar{G} \subseteq \overline{\mathbb{C}} \backslash \operatorname{supp}(\mu)$ and $\operatorname{cap}(\partial G) > 0$. Define $\nu_n := \mu_n|_{\bar{G}}$ and $\mu_n^* := \mu_n - \nu_n$. We have $\nu_n \overset{*}{\to} 0$ as $n \to \infty$, and it is not difficult to verify that $p(\mu_n; z) - g_G(\nu_n; z) \to p(\mu; z)$ locally uniformly for $z \in G$ as $n \to \infty$. From Lemma 2.14 it then follows that $p(\mu_n; \cdot)$ converges to $p(\mu; \cdot)$ in capacity in G as $n \to \infty$. Since each component of $\overline{\mathbb{C}} \backslash \operatorname{supp}(\mu)$ can be exhausted by domains G of the considered type, the corollary is proved. ∎

Remark. We note that in the corollary convergence in capacity holds in general not on subsets of $\operatorname{supp}(\mu)$. This will be shown by the following example: Consider the sequence of polynomials $p_n(z) := z^n - 1$, $n = 1, 2, \ldots$. We have

$$\mu_{\text{n}} := \frac{1}{n} \quad \nu_{p_n} \overset{*}{\to} \lambda \qquad \text{as} \quad n \to \infty, \tag{2.44}$$

where λ is the uniformly distributed probability measure on $\partial \mathbb{D}$. Set $f_n := p(\mu_n; \cdot)$, $f := p(\lambda; \cdot) = -\log \max(1, |\cdot|)$, and $V := \overline{\mathbb{D}(0, 2)}$. Then it is not too difficult to verify that

$$\lim_{n \to \infty} \operatorname{cap}\{z \in V \mid |(f_n - f)(z)| > \varepsilon\} = e^{-\varepsilon}, \qquad \varepsilon > 0. \tag{2.45}$$

Actually, one can show that

$$e^{-\varepsilon} \leqslant \operatorname{cap}\{z \in V \mid |(f_{\boldsymbol{n}} - f)(z)| > \varepsilon\} \leqslant e^{-\varepsilon + 1/n} \tag{2.46}$$

for $\varepsilon > 0$ sufficiently small and $n \in \mathbb{N}$ sufficiently large.

Proof of Lemma 2.14. Choose $\varepsilon > 0$ and a compact set $V \subseteq G \cap \mathbb{C}$. There exists a compact set $V^* \subseteq G$ with $V \subseteq \operatorname{Int}(V^*)$. Define

$$V_n^* := \{z \in V^* \mid |(f_n - f)(z)| > \varepsilon\}. \tag{2.47}$$

In a first step we assume that

$$V^* \subseteq \overline{\mathbb{D}}. \tag{2.48}$$

Since $\nu_n(V^*) \to 0$, we have

$$\lim_{n \to \infty} [g_G(\nu_n|_{V^*}; z) - p(\nu_n|_{V^*}; z)] = 0 \tag{2.49}$$

uniformly for $z \in V$. Because of the assumptions (i), (ii), and (iii) for any $0 < \delta < \varepsilon/4$ there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} \|\hat{f}_n - f\|_V &< \delta, \\ \|g_G(v_n|_{G\setminus V^*}; \cdot)\|_V &< \delta \\ \|g_G(v_n|_{V^*}; \cdot) - p(v_n|_{V^*}; \cdot)\|_V &< \delta, \end{aligned} \tag{2.50}$$

for all $n \geqslant n_0$, where $\|\cdot\|_V$ denotes the sup-norm on V. From (2.42), (2.47), and (2.50) it follows that

$$\begin{aligned} |p(v_n|_{V^*}; z)| &= p(v_n|_{V^*}; z) - g_G(v_n|_{V^*}; z) - g_G(v_n|_{G\setminus V^*}; z) \\ &\quad + (f_n - f)(z) + (f - \hat{f}_n)(z) \\ &> \varepsilon - \|\hat{f}_n - f\|_V - \|g_G(v_n|_{G\setminus V^*}; \cdot)\|_V \\ &\quad - \|g_G(v_n|_{V^*}; \cdot) - p(v_n|_{V^*}; \cdot)\|_V \\ &> \varepsilon - 3\delta > \delta \end{aligned} \tag{2.51}$$

for all $z \in V_n^* \cap V$. From assumption (i) and the principle of descent (Lemma 2.1) we know that

$$\liminf_{n\to\infty} p(v_n|_{V^*}; z) \geqslant 0 \tag{2.52}$$

uniformly for $z \in V$. Hence, there exists $n_1 \in \mathbb{N}$ such that $p(v_n|_{V^*}; z) > -\delta$ for all $z \in V^*$ and $n \geqslant n_1$. Therefore, it follows from (2.51) that

$$p(v_n|_{V^*}; z) > \delta \qquad \text{for} \quad z \in V_n^* \cap V \text{ and } n \geqslant n_1. \tag{2.53}$$

Comparing the two functions

$$p\left(\frac{v_n|_{V^*}}{v_n(V^*)}; \cdot\right) \qquad \text{and} \qquad -g_{\overline{\mathbb{C}}\setminus(V_n^*\cap V)}(z, \infty) - \log \operatorname{cap}(V_n^* \cap V)$$

yields the estimate

$$\operatorname{cap}(V_n^* \cap V) \leqslant e^{-\delta/v_n(V^*)}, \qquad n \geqslant n_1, \tag{2.54}$$

where we have used the principle of domination (Lemma 2.4), and representation (2.21) in Lemma 2.6, together with the special expression for c_∞ in the case of $V_n^* \cap V \subseteq \overline{\mathbb{D}}$. Note that the equilibrium distribution $\omega_{V_n^* \cap V}$ is of finite energy if $\operatorname{cap}(V_n^* \cap V) > 0$. Since $v_n(V^*) \to 0$, (2.54) implies the convergence in capacity.

If assumption (2.48) does not hold true, we can shrink the whole problem by the mapping $z \mapsto rz$, $0 < r < 1$. Because of the homogenity of the capacity, i.e. $\operatorname{cap}(rS) = r\operatorname{cap}(S)$, the right-hand side of (2.54) changes only by the factor $1/r$, which proves the lemma in the general case. ▮

LEMMA 1.16. *If the two sequences $\{f_n\}$ and $\{g_n\}$ converge in capacity to f and g, respectively, in the domain $G \subseteq \overline{\mathbb{C}}$ as $n \to \infty$, then also the sum $\{f_n + g_n\}$ converges in capacity to $f + g$ in G.*

Proof. Let $V \subseteq G \cap \mathbb{C}$ be a compact set, $\varepsilon > 0$, and define

$$\begin{aligned} V_n^1 &:= \left\{ z \in V \mid |(f_n - f)(z)| > \frac{\varepsilon}{2} \right\}, \\ V_n^2 &:= \left\{ z \in V \mid |(g_n - g)(z)| > \frac{\varepsilon}{2} \right\}, \\ V_n &:= \{ z \in V \mid |[(f_n + g_n) - (f + g)](z)| > \varepsilon \}. \end{aligned} \tag{2.55}$$

It is immediate that

$$V_n \subseteq V_n^1 \cup V_n^2. \tag{2.56}$$

The proof would be finished here if the capacity were an additive set function, but unfortunately this is not the case. However, we know the following subadditivity (cf. [12, Chap. V]): Let N compact subsets $W_j \subseteq W \subseteq \mathbb{C}$ be given with W a compact set and $\operatorname{cap}(W_j) \leqslant a$ for $j = 1, ..., N$. Then

$$\operatorname{cap}(W_1 \cup \cdots \cup W_N) \leqslant a^{1/N} \operatorname{diam}(W)^{1 - 1/N}. \tag{2.57}$$

Applying (2.57) to (2.56) yields

$$\operatorname{cap}(V_n) \leqslant \operatorname{cap}(V_n^1 \cup V_n^2) \leqslant \sqrt{\max(\operatorname{cap}(V_n^1), \operatorname{cap}(V_n^2))}\,\sqrt{\operatorname{diam}(V)}, \tag{2.58}$$

which proves the lemma. ▮

3. SOME AUXILIARY LEMMAS.

Theorem 1.8 plays a key role in the proofs of all other results given in Section 1. Among other things, the theorem is concerned with the asymptotic distribution of poles of the Padé approximants $[m_j/n_j]$ as $j \to \infty$. The theorem itself will be proved indirectly by showing that the equilibrium distribution ω of the set F is the only possibility for an asymptotic distribution for any infinite subsequence of Padé approximants $\{[m_j/n_j]\}$ (F is the complement of the convergence domain D). The basis of the indirect proof is the existence of a special measure μ. The existence of this measure is proved in the next lemma. The second lemma contains a similar result, which is needed in the proof of the second statement of Theorem 1.2. The section is closed with two technical lemmas about the asymptotic behavior of sequences of polynomials.

The objects D, F, F_0, $\omega = \omega_F$, Φ, U, and $L(x, \delta)$ have the same meaning as in the last section. By $c_1, c_2, \ldots$ we denote constants.

LEMMA 3.1. *Let $D \subseteq \overline{\mathbb{C}}$ be a symmetric domain as introduced in Definition 1.3 and let $F_1 \subseteq F$ be a compact set with $F_0 \subseteq F_1$, $\operatorname{cap}(F_1) = 0$. For any measure $\mu \geqslant 0$ with $|\mu| \leqslant 1$ and $\mu \neq \omega$ there exists a measure $\nu \geqslant 0$ such that $\|\nu\| < 1$, $\operatorname{supp}(\nu)$ is contained in a preassigned neighborhood of F, and there exists $x \in F \setminus F_1$ and $\delta_0 > 0$ with $\overline{L(x, \delta_0)} \subseteq U \setminus F_1$ such that*

(i) $p(\nu + \mu; z) = \infty$ *for* $z \in F_1$,

(ii) $\min_{z \in F \cap \overline{L(x, \delta)}} p(\nu + \mu; z) < \min_{z \in F \setminus L(x, \delta)} p(\nu + \mu; z)$ *for all* $0 < \delta \leqslant \delta_0$,

(iii) $p(\nu + \mu; z) = p(\nu + \mu; \Phi(z))$ *for* $z \in L(x, \delta_0)$,

(iv) $\mu|_{L(x, \delta_0)} = \Phi(\nu|_{L(x, \delta_0)})$.

Proof. We shall proceed in two stages. In the first one it will be shown that there exist two measures $\nu_1, \nu_2 \geqslant 0$ with $\|\nu_1 + \nu_2\| < 1$, $\operatorname{supp}(\nu_1) \subseteq U \cup F$, and $\operatorname{supp}(\nu_2) \subseteq F$ such that

(a) $p(\nu_2; z) = \infty$ for all $z \in F_1$, (3.1)

(b) $p(\mu + \nu_1 + \nu_2; \cdot\,)$ has a unique minimum on F at a certain point $x \in F \setminus F_1$,

(c) $p(\mu + \nu_2; z) = p(\nu_1; z) + c_1$ for all $z \in L(x, \delta_0)$ with $\delta_0 > 0$ such that $\operatorname{supp}(\nu_2) \cap \overline{L(x, \delta_0)} = \varnothing$ and $c_1 \in \mathbb{R}$, (3.2)

(d) $p(\nu_1; z) = p(\nu_1; \Phi(z))$ for all $z \in U \setminus L(x, \delta_1)$, where $\delta_0 < \delta_1$ with $\overline{L(x, \delta_1)} \subseteq U \setminus F_1$ and $\operatorname{supp}(\nu_1) \subseteq F \cup L(x, \delta_1)$. (3.3)

The measures ν_1 and ν_2 will be constructed in a chain of modifications of the given measure μ. We shall describe this procedure step by step.

Step 1. If $\|\mu\| = 1$ and $\operatorname{supp}(\mu) \subseteq F$, then it follows from the assumption $\mu \neq \omega$ that for any constant $c \in \mathbb{R}$ we have

$$p(\mu; z) \neq c \qquad \text{for more than quasi every } z \in F. \tag{3.4}$$

Hence, there exists a constant $c_2 \in \mathbb{R}$ such that for both sets $E_1 := \{z \in F \mid p(\mu; z) < c_2\}$ and $O_1 := \{z \in \overline{\mathbb{C}} \mid p(\mu; z) > c_2\}$ we have

$$\operatorname{cap}(E_1) > 0 \qquad \text{and} \qquad \operatorname{cap}(F \cap O_1) > 0. \tag{3.5}$$

From the lower semicontinuity of $p(\mu; \cdot)$ it follows that O_1 is open. Let $\{O_{1j}\}$ be the set of components of O_1. Each O_{1j} is simply connected because of the superharmonicity of $p(\mu; \cdot)$. Since $O_1 \neq \emptyset$ there exists at least one component, the component O_{11}. From the definition it follows that $O_{11} \cap \operatorname{supp}(\mu) \neq \emptyset$. Let μ_1 be the balayage measure resulting from sweeping μ out of the domain O_{11} (cf. Lemma 2.3). Then we have

$$p(\mu_1; z) = p(\mu; z) + c_3 \qquad \text{for all} \quad z \in \overline{\mathbb{C}} \setminus O_{11}. \tag{3.6}$$

Note that $\overline{\mathbb{C}} \setminus O_{11}$ has no irregular points (with respect to the Dirichlet Problem in O_{11}) since O_{11} is simply connected. Since $O_{11} \cap \operatorname{supp}(\mu) \neq \emptyset$ and since ∂O_{11} is not contained in F, which can be obtained in any case by choosing c_2 in the definition of the sets E_1 and O_1 appropriately, we have $\operatorname{supp}(\mu_1) \backslash F \neq \emptyset$ (cf. Lemma 2.3). If one of the two assumptions $|\mu| = 1$ and $\operatorname{supp}(\mu) \subseteq F$ in step 1 is not true, then we define $\mu_1 := \mu$.

Step 2. If $\operatorname{supp}(\mu_1) \subseteq F$, then we have $\mu_1 = \mu$ and $|\mu| < 1$. In this case we set $\mu_2 := \mu_1 = \mu$, in all other cases we have $\operatorname{supp}(\mu_1) \backslash F \neq \emptyset$, and the measure μ_1 will be modified in the present step in the following way: Define $\tilde{E}_2 = \tilde{E}_2\ (\delta_1) := \{z \in \mathbb{C} \,|\, \operatorname{dist}(z, F) \leqslant \delta_1\}$, $\delta_1 > 0$, and let E_2 be the polynomial convex hull of $\tilde{E}_2$, i.e. E_2 is $\tilde{E}_2$ together with all bounded components of $\mathbb{C} \backslash \tilde{E}_2$. Since $\operatorname{supp}(\mu_1) \backslash F \neq \emptyset$, we have $\operatorname{supp}(\mu_1) \backslash E_2 \neq \emptyset$ for $\delta_1 > 0$ sufficiently small. Let μ_2 be the balayage measure resulting from balayage with mass reduction of μ_1 out of the domain $\overline{\mathbb{C}} \backslash E_2$, as introduced in Lemma 2.7. Thus, we have

$$\|\mu_2\| < 1 \tag{3.7}$$

and

$$p(\mu_2; z) = p(\mu; z) + c_4 \qquad \text{for} \quad z \in E_2 \setminus O_{11}. \tag{3.8}$$

Note that E_2 has no irregular points. In the deduction of (3.8) the identities (3.6) and (2.23) have been used. After the first two modification steps we see that (3.7) and (3.8) hold independently of the different cases that were considered in the definition of μ_1 and μ_2.

Step 3. From Lemma 2.11 we know that there exists a probability measure σ_1 on F_1 such that

$$\mathrm{p}(\sigma_1; z) = \infty \qquad \text{for all} \quad z \in F_1. \tag{3.9}$$

We choose δ_2 such that

$$0 < 2\delta_2 < 1 - \|\mu_2\|. \tag{3.10}$$

Because of its lower semicontinuity the potential $p(\mu_2+\delta_2\sigma_1;\cdot)$ assumes its minimum on F. If $\delta_2>0$ is sufficiently small, then because of (3.9), (3.8), and definition of O_1, the minimum is assumed at a point $x\in F\setminus(F_1\cup\overline{O_{11}})$. From Lemma 2.12 we know that there exists a probability measure σ_2 with $\operatorname{supp}(\sigma_2)\subseteq F\setminus\{x\}$ and that $p(\mu_2+\delta_2\sigma_1+\delta_3\sigma_2;\cdot)$ has its unique minimum at the same point $x\in F\setminus(F_1\cup\overline{O_{11}})$ and the number δ_3 satisfies

$$0<2\delta_3<1-\|\mu_2\|-2\delta_2. \tag{3.11}$$

Because of (3.8) and since $p(\mu;z)>p(\mu_2;z)-c_4$ for $z\in E_2\cap\overline{O_{11}}$, it follows that also $p(\mu+\delta_2\sigma_1+\delta_3\sigma_2;\cdot)$ assumes its unique minimum on F at $x\in F\setminus(F_1\cup\overline{O_{11}})$. We define

$$\nu_2:=\delta_2\sigma_1+\delta_3\sigma_2. \tag{3.12}$$

From (3.10) and (3.11) it follows that

$$\|\mu_2\|+2\delta_2+2\delta_3<1. \tag{3.13}$$

Step 4. We now come to the most difficult part of the analysis. Define $\mu_3:=\mu_2+\nu_2$ and $E_3=E_3(\delta_4):=\{z\in\overline{\mathbb{C}}\mid p(\mu_3;z)-p(\mu_3;x)\leqslant-\|\mu_3\|\, g_D(z,\infty)+\delta_4\}$, $\delta_4>0$. From Lemma 2.9 and the structure of F given in (1.24) we deduce that $g_D(z,\infty)$ is continuous in $\mathbb{C}\setminus F_0$. Therefore it follows from the lower semicontinuity of $p(\mu_3;\cdot)$ that E_3 is closed in $\overline{\mathbb{C}}\setminus F_1$. It is immediate that $E_3(\delta_4)$ depends monotonically on δ_4, and $x\in E_3\ (\delta_4)$ for all $\delta_4\geqslant 0$. The uniqueness of the minimum of $p(\mu_3;\cdot)$ on F at x implies that $\bigcap_{\delta_4>0}E_3(\delta_4)=\{x\}$. Let $\delta_5>0$ be such that $\overline{L(x,\delta_5)}\subseteq U\setminus F_1$. For $\delta_4>0$ sufficiently small we have $E_3(\delta_4)\subseteq L(x,\delta_5)$. Let O_2 be the component of the open set $L(x,\delta_5)\setminus E_3(\delta_4)$ with $\partial L(x,\delta_5)\subseteq\partial O_2$, and set $E_4:=L(x,\delta_5)\setminus O_2$. It is immediate that $x\in E_4$, but it is necessary to show that the stronger assertion

$$x\in\operatorname{Int}(E_4) \tag{3.14}$$

holds true. Indeed, let $x\in J_j$ and let U_{j+}, U_{j-} be the two subdomains of $U_j\setminus J_j$, i.e. $U_j=U_{j+}\cup J_j\cup U_{j-}$. Let the function h_1 be defined by $h_1:=g_D(\cdot,\infty)$ on $U_{j+}\cup J_j$ and by $h_1:=-g_D(\cdot,\infty)$ on U_{j-}. Because of the symmetry (1.14), which has been assumed in Definition 1.3, h_1 is harmonic in U_j, and consequently $h_2:=p(\mu_3;\cdot)-p(\mu_3;x)+\|\mu_3\|\,h_1$ is superharmonic in U_j. Define $\tilde{O}_3:=\{z\in L(x,\delta_5)\ \ |h_2(z)>\delta_4\}$. Note that $\overline{L(x,\delta_5)}\subseteq U_j$. Let O_3 be the component $\tilde{O}_3$ with $\partial O_3\cap\partial L(x,\delta_5)\neq\emptyset$. From the definition of the sets E_3, O_2, and O_3 it follows that $O_3\cap(U_{j+}\cup J_j)$-$=O_2\cap(U_{j+}\cup J_j)$. The function h_2 is superharmonic, and $h_2(x)=0<\delta_4$. From Lemma 2.5 we therefore can deduce that $x\notin\overline{O_3}$. Next we repeat the arguments with interchanging the role of U_{j+} and U_{j-}, i.e. h_3 is now defined by $h_3 := g_D(\cdot, \infty)$ on $U_{j-} \cup J_j$ and $h_3 := -g_D(\cdot, \infty)$ on U_{j+}, and further $h_4 := p(\mu_3; \cdot) - p(\mu_3; x) + \|\mu_3\| h_3$. The open sets $\tilde{O}_4$ and O_4 are defined like $\tilde{O}_3$ and O_3, only that the function h_4 is used instead of h_2. With the same arguments as before we conclude that $x \notin \overline{O_4}$. From the definition of the sets O_2, O_3, and O_4 we deduce that $O_2 = O_3 \cup O_4 = L(x, \delta_5) \backslash E_4$. This implies that $x \notin \overline{O_2} = \overline{O_3} \cup \overline{O_4}$, and thus proves (3.14).

We define

$$h_5(z) := \begin{cases} -\|\mu_3\| \, g_D(z, \infty) + \delta_4 & \text{for} \quad z \in \overline{\mathbb{C}} \backslash \mathrm{Int}(E_4) \\ p(\mu_3; z) - p(\mu_3; x) & \text{for} \quad z \in \mathrm{Int}(E_4). \end{cases} \tag{3.15}$$

The function is superharmonic in $\mathbb{C}$ and harmonic in $\mathbb{C} \backslash (F \cup E_4)$. Indeed, the lower semicontinuity can be verified by one-sided approximation with continuous functions. The mean-value inequality can be verified directly on the basis of (3.15). From Lemma 2.13 we know that there exists a positive measure ν_1 with $\|\nu_1\| = \|\mu_3\|$, $\mathrm{supp}(\nu_1) \subseteq F \cup E_4 \subseteq F \cup L(x, \delta_5)$, and

$$\mathrm{p}(\nu_1; z) = h_5(z) + c_5 \qquad \text{for} \quad z \in \mathbb{C}. \tag{3.16}$$

Step 5. From (3.12), (3.13), $\|\nu_1\| = \|\mu_3\|$, and $\mu_3 = \mu_2 + \nu_2$, it follows that $\|\nu_1 + \nu_2\| < 1$. From (3.14) we deduce that there exists $\delta_6 > 0$ with $\overline{L(x, \delta_6)} \subseteq \mathrm{Int}(E_4)$. From (3.15), (3.16), (3.8), and $\mu_3 = \mu_2 + \nu_2$ it follows that

$$p(\nu_1; z) = p(\mu + \nu_2; z) + c_6 \qquad \text{for all} \quad z \in L(x, \delta_6). \tag{3.17}$$

Assertion (a) is a consequence of (3.9) and (3.12). Assertion (b) is a consequence of the unique minimum of $p(\mu + \nu_2; \cdot)$ at x together with the identities (3.15) and (3.16), which imply that $p(\nu_1; \cdot)$ has a minimum on $F \backslash F_1$ at the point x. Assertion (c) follows from (3.17) with the choice $\delta_0 := \delta_6$. Assertion (d) is a consequence of (3.15), (3.16) and the Φ-symmetry of $g_D(\cdot, \infty)$ if one chooses $\delta_1 := \delta_4$. Note that on $\overline{\mathbb{C}} \backslash L(x, \delta_4)$ we have $p(\nu_1; \cdot) = -\|\mu_3\| \, g_D(\cdot, \infty) + c_5 + \delta_4$.

Next, we come to the second stage of the proof. Define

$$h_6(z) := \begin{cases} p(\nu_1; z) & \text{for} \quad z \in \overline{\mathbb{C}} \backslash U \\ p(\nu_1; \Phi(z)) & \text{for} \quad z \in U. \end{cases} \tag{3.18}$$

From assertion (d) we deduce that h_6 is superharmonic on $F \cup \overline{L(x, \delta_1)}$ and is harmonic outside of $F \cup \overline{L(x, \delta_1)}$ since it follows from (3.3) that both functions on the right-hand side of (3.18) coincide on $U \backslash L(x, \delta_1)$. From

Lemma 2.13 we therefore know that there exists a positive measure ν_3 with $\operatorname{supp}(\nu_3) \subseteq F \cup \overline{L(x, \delta_1)}$, $\|\nu_3\| = \|\nu_1\|$, and

$$\mathrm{p}(\nu_3; z) = h_6(z) + c_7 \qquad \text{for all} \quad z \in \mathbb{C}. \tag{3.19}$$

We define

$$\nu := \nu_2 + \nu_3. \tag{3.20}$$

Assertion (i) of the lemma then follows from assertion (a) and (3.20). From assertion (b) together with (3.18), (3.19), and (3.20) it follows that $p(\nu + \mu; \cdot)$ has its unique minimum at the point x, which implies assertion (ii) of the lemma. Assertion (iii) follows from assertion (c) together with (3.18) and (3.19). From assertion (c) it further follows that

$$\mu|_{L(x, \delta_0)} = \nu_1|_{L(x, \delta_0)}. \tag{3.21}$$

Note that $\operatorname{supp}(\nu_2) \cap L(x, \delta_0) = \emptyset$. From (3.18) and (3.19) we deduce that

$$p(\nu_3; z) = p(\nu_1; \Phi(z)) + c_7 \qquad \text{for all} \quad z \in U, \tag{3.22}$$

which implies that

$$\nu_3|_{L(x, \delta_0)} = \Phi(\nu_1|_{L(x, \delta_0)}). \tag{3.23}$$

Since $\operatorname{supp}(\nu_2) \cap \overline{L(x, \delta_0)} = \emptyset$, (3.20), (3.21), and (3.23) imply assertion (iv). The proof of Lemma 3.1 is completed. ∎

With the proof of Lemma 3.1 the perhaps technically most difficult part of the paper has been completed. The next lemma should be seen as a variation of Lemma 3.1.

LEMMA 3.2. *Let $D \subseteq \overline{\mathbb{C}}$ be a symmetric domain as introduced in Definition* 1.3. *For any compact set $F_1 \subseteq F$ with $F_0 \subseteq F_1$,* $\operatorname{cap}(F_1) = 0$, *and any $\varepsilon > 0$, there exists a measure $\nu \geqslant 0$ such that* $\|\nu\| \leqslant \varepsilon$, $\operatorname{supp}(\nu)$ *compact, and there exists $x \in F \setminus F_1$ and $\delta_0 > 0$ with $\overline{L(x, \delta_0)} \subseteq U \setminus F_1$ such that*

(i) $p(\omega + \nu; z) = \infty$ *for all* $z \in F_1$,

(ii) $\min_{z \in F \cap \overline{L(x, \delta)}} p(\omega + \nu; z) < \min_{z \in F \setminus L(x, \delta)} p(\omega + \nu; z)$ *for all* $0 < \delta \leqslant \delta_0$,

(iii) $p(\omega + \nu; z) = p(\omega + \nu; \Phi(z))$ *for all* $z \in L(x, \delta_0)$,

(iv) $\nu|_{\overline{L(x, \delta_0)}} = 0.$

Proof. The structure of the proof is very similar to that of Lemma 3.1; the arguments are often only a special case of those applied in the earlier proof. Again we proceed in two stages. In the first stage we show that there exist two positive measures $\nu_1, \nu_2 \geqslant 0$ with $\|\nu_1 + \nu_2\| \leqslant \varepsilon$, $\mathrm{supp}(\nu_1) \subseteq \mathrm{U} \cup \mathrm{F}$, and $\mathrm{supp}(\nu_2) \subseteq \mathrm{F}$ such that

(a) $p(\nu_2; z) = \infty$ for all $z \in F_1$, (3.24)

(b) $p(\nu_1 + \nu_2; \cdot)$ has a unique minimum at a certain point $x \in F \backslash F_1$,

(c) $p(\nu_1; z) = p(\nu_2; z) + c_1$ for all $z \in L(x, \delta_0)$ and for some $\delta_0 > 0$, $c_1 \in \mathbb{R}$, (3.25)

(d) $p(\nu_1; z) = p(\nu_1; \Phi(z))$ for all $z \in U \backslash L(x, \delta_1)$ and for some $\delta_1 > \delta_0$ with $\overline{L(x, \delta_1)} \subseteq \mathrm{U} \backslash \mathrm{F}_1$, (3.26)

(e) $\nu_1|_{L(x, \delta_0)} = \nu_2|_{L(x, \delta_0)} = 0.$ (3.27)

Let σ_1 be a probability measure with $\mathrm{supp}(\sigma_1) \subseteq F_1$ such that (3.9) holds true (cf. Lemma 2.11), and let x be a point on F, where $p(\sigma_1; \cdot)$ assumes its minimum on F. Because of (3.9) we have $x \in F \backslash F_1$. From Lemma 2.12 we know that there exists a probability measure σ_2 with $\mathrm{supp}(\sigma_2) \subseteq F \backslash \{x\}$ such that $p(\sigma_1 + \sigma_2; \cdot)$ has its unique minimum on F at $x \in \mathrm{F} \backslash \mathrm{F}_1$. We define

$$\nu_2 := \frac{\varepsilon}{4} (\sigma_1 + \sigma_2). \tag{3.28}$$

Hence, $\|\nu_2\| \leqslant \varepsilon/2$ and $p(\nu_2; \cdot)$ assumes its unique minimum at $x \in F \backslash F_1$.

As in the text before (3.14) we can show that there exist $\delta_2, \delta_3 > 0$ such that the closed set $E_1 = E_1(\delta_2) := \{z \in \overline{\mathbb{C}} \mid p(\nu_2; z) - p(\nu_2; x) \leqslant -\|\nu_2\| \, g_D(z, \infty) + \delta_2\} \subseteq L(x, \delta_3)$ and $\overline{L(x, \delta_3)} \subseteq U \backslash F_1$. Let O_1 be the component of the open set $L(x, \delta_3) \backslash E_1$ with $\partial L(x, \delta_3) \subseteq \partial O_1$, and define $E_2 := L(x, \delta_3) \backslash O_1$. As in the text after (3.14), we then can show that

$$x \in \mathrm{Int}(E_2). \tag{3.29}$$

Hence, there exists $\delta_4 > 0$ with $\overline{L(x, \delta_4)} \subseteq \mathrm{Int}(\mathrm{E}_2)$.

Analogously to (3.15) we define

$$h_1(z) := \begin{cases} -\|\nu_2\| \, g_D(z, \infty) + \delta_2 & \text{for} \quad z \in \overline{\mathbb{C}} \backslash \mathrm{Int}(E_2) \\ p(\nu_2; z) - p(\nu_2; x) & \text{for} \quad z \in \mathrm{Int}(E_2), \end{cases} \tag{3.30}$$

and as after (3.15) we deduce from the superharmonicity of h_1 that there exists a positive measure ν_1 with $\|\nu_1\| = \|\nu_2\| \leqslant \varepsilon/2$, $\operatorname{supp}(\nu_1) \subseteq F \cup L(x, \delta_3)$, $c_2 \in \mathbb{R}$, and

$$p(\nu_1; z) = h_1(z) + c_2 \qquad \text{for all} \quad z \in \mathbb{C}. \tag{3.31}$$

Thus, we have $\|\nu_1 + \nu_2\| \leqslant \varepsilon$. From (3.30) and (3.31) it follows that

$$p(\nu_2; z) = p(\nu_1; z) + c_3 \qquad \text{for all} \quad z \in L(x, \delta_4). \tag{3.32}$$

If $\delta_4 > 0$ is chosen sufficiently small, then $\nu_2|_{L(x, \delta_4)} = 0$ since $\operatorname{supp}(\nu_2) \subseteq F \backslash \{x\}$. From (3.32) it then follows that also $\nu_1|_{L(x, \delta_4)} = 0$, which implies assertion (e) if we choose $\delta_0 := \delta_4$. With the same choice of δ_0 and $c_1 = c_3$, assertion (c) follows (3.32). As in the proof of Lemma 3.1, assertion (a) is a consequence of (3.9) and (3.28). Assertion (b) follows from the fact that $p(\nu_2; \cdot)$ assumes its minimum on F uniquely at x, and that because of (3.30) and (3.31) the potential $p(\nu_1; \cdot)$ assumes its minimum on $F \backslash F_1$ at the same point x. Assertion (d) follows from (3.30), (3.31), and the Φ-symmetry of $g_D(\cdot, \infty)$.

We come to the second stage of the proof, which is again very analogous to the second stage in the proof of Lemma 3.1. Define

$$h_2(x) := \begin{cases} p(\nu_1; z) & \text{for} \quad z \in \overline{\mathbb{C}} \backslash U \\ p(\nu_1; \Phi(z)) & \text{for} \quad z \in U. \end{cases} \tag{3.33}$$

As after (3.18) it follows from the superharmonicity of h_2 that there exists a positive measure with $\|\nu_3\| = \|\nu_1\|$, $\operatorname{supp}(\nu_3) \subseteq F \cup L(x, \delta_3)$, and

$$p(\nu_3; z) = h_2(z) + c_4 \qquad \text{for all} \quad z \in \mathbb{C}, \tag{3.34}$$

and as before we define

$$\nu := \nu_2 + \nu_3. \tag{3.35}$$

We have $\|\nu\| \leqslant \varepsilon$. Assertion (i) of the lemma follows from assertion (a) and (3.35). We deduce from Lemmas 2.6 and 2.9 that there exists $c_5 \in \mathbb{R}$ with $p(\omega; \cdot) = c_5$ on $F \backslash F_0$ and $p(\omega; \cdot) \leqslant c_5$ on F_0. Assertion (ii) therefore follows from assertions (b) and (a). Together with (3.33), (3.34), and the Φ-symmetry of $p(\omega; \cdot)$, assertion (c) implies assertion (iii) of the lemma. The Φ-symmetry of $p(\omega; \cdot)$, is an immediate consequence of Lemmas 2.6 and 2.8. At last, assertion (iv) follows from assertion (e) together with (3.35), (3.34), and (3.33). This completes the proof of Lemma 3.2. ∎

In Section 4 at several places we will be concerned with sequences of polynomials that are asymptotically Φ-symmetric in the nth root sense, and it will be necessary to modify them by polynomial factors in such a way that the modified sequence is asymptotically Φ-symmetric in a stronger sense. For this procedure we need the next lemma. Its statement will be prepared by some considerations concerning Φ-symmetry.

It is an immediate consequence of the Φ-symmetry introduced in Definition 2.2 that for a function f analytic in U the function $\log|f|$ is Φ-symmetric if and only if the quotient

$$\frac{f}{\overline{f\circ\Phi}} \equiv \mathrm{c} \qquad \text{on } U \tag{3.36}$$

with $c\in\mathbb{C}$, $|c|=1$. For any function f analytic in U, not necessarily Φ-symmetric, the quotient (3.36) is meromorphic in U. If the quotient is analytic in U, then it is also different from zero in U, and the set of zeros of f in U is Φ-symmetric.

Lemma 3.3. *Let $D\subseteq\overline{\mathbb{C}}$ be a symmetric domain as introduced in Definition 1.3, and let $F_1\subseteq F$ be compact with $F_1\supseteq F_0$ and $\mathrm{cap}(F_1)=0$. Further, let $x\in F\backslash F_1$ and $\delta_0>0$ be such that $\overline{L(x,\delta_0)}\subseteq U\mathrm{U}\backslash F$. We consider a sequence of polynomials $p_n\in\mathcal{P}_n$, $n=1,2,\ldots$, with $p_n(x)=1$, $Z_n:=Z(p_n)$ the set of zeros of p_n, and assume $Z_n|_{L(x,\delta_0)}$ to be $\Phi-$symmetric for $n\in\mathbb{N}$ sufficiently large, i.e.,*

$$Z_{\mathrm{n}}|_{L(x,\delta_0)}=\Phi(Z_n|_{L(x,\delta_0)}) \qquad \textit{for} \quad n\geqslant n_0. \tag{3.37}$$

If

$$\lim_{n\to\infty}\left(\frac{p_n(z)}{\overline{p_n(\Phi(z))}}\right)^{1/\mathrm{n}}=1 \tag{3.38}$$

locally uniformly for $z\in L(x,\delta_0)$, then there exists $\delta_1>0$ and a sequence of polynomials $h_n\in\mathcal{P}_{\mathrm{k_n}}$, $n=1,2,\ldots$, with $h_n(x)=1$, and numbers $k_n\in\mathbb{N}$ such that

$$\text{(i)}\quad \lim_{n\to\infty}(k_n/n)=0 \tag{3.39}$$

$$\text{(ii)}\quad \limsup_{n\to\infty}|h_n(z)|^{1/n}\leqslant 1 \textit{ locally uniformly for } z\in\mathbb{C}, \tag{3.40}$$

and

$$\text{(iii)}\quad \lim_{n\to\infty}((h_n p_n)(z)/(h_n p_n)(\Phi(z)))=1 \textit{ locally uniformly for } z\in L(x,\delta_1). \tag{3.41}$$

Remarks. (1) Assumption (3.38) means that $(1/n) \log |p_n|$ is asymptotically Φ-symmetric in $L(x, \delta_0)$. The lemma shows that multiplication by h_n produces a sequence $\log |h_n p_n|$ that is also asymptotically Φ-symmetric, but now without a division by n.

(2) While the limit (3.38) holds in $L(x, \delta_0)$ the limit (3.41) holds in general only in a smaller Φ-disc $L(x, \delta_1)$. A more sophisticated construction of the polynomials h_n could have extended limit (3.41) to the original Φ-disc $L(x, \delta_0)$.

Proof. Assume that $0 < \delta_1 < \delta_2 < \delta_0$. The numbers δ_1 and δ_2 will be fixed below. Because of (3.38) there exists a sequence $\{\varepsilon_n > 0\}$ with $\varepsilon_n \to 0$ such that

$$(1+\varepsilon_{\mathrm{n}})^{-2\mathrm{n}} \leqslant \left| \frac{p_{\mathrm{n}}}{\overline{p_n \circ \Phi}} (z) \right| \leqslant (1-\varepsilon_n)^{-2n} \tag{3.42}$$

for all $z \in \overline{L(x, \delta_2)}$ and $n \geqslant n_0$. There exists a sequence $k_n \in \mathbb{N}$, $n \in \mathbb{N}$, such that

$$\frac{k_n}{n} \to 0, \ \frac{k_n}{\varepsilon_n n} \to \infty, \ k_n \to \infty \qquad \text{as} \quad n \to \infty. \tag{3.43}$$

From (3.43) we deduce that for any $m \in \mathbb{N}$ and $0 < \eta_2 < \eta_1 < 1$ and sufficiently large $n \in \mathbb{N}$ we have

$$(1-\eta_1)^{k_n} (1+\varepsilon_n)^{mn} \leqslant (1-\eta_2)^{k_n}. \tag{3.44}$$

Indeed, this follows from

$$\begin{aligned}[(1-\eta_1)(1+\varepsilon_n)^{mn/k_n}]^{k_n} &= \left[(1-\eta_1) \left(1 + m \left(\frac{\varepsilon_n n}{k_n} \right) \frac{k_n}{mn} \right)^{mn/k_n} \right]^{kn} \\ &\leqslant [(1-\eta_1)(e^0+\varepsilon)]^{k_n} \leqslant (1-\eta_2)^{k_n}. \end{aligned} \tag{3.45}$$

The estimates in (3.45) hold for any $\varepsilon > 0$ and $n \in \mathbb{N}$ sufficiently large. We define

$$g_n(z) := \sqrt{\frac{\overline{p_n(\Phi(z))}}{p_n(z)}} \qquad \text{for} \quad z \in L(x, \delta_0), \ n \geqslant n_0. \tag{3.46}$$

The square root is defined for $n \geqslant n_0$ since because of (3.37) its argument is analytic and different from 0. It is immediate that

$$\overline{g_{\mathrm{n}}(\Phi(z))} = \frac{1}{g_n(z)}, \qquad \frac{(p_n g_n)(z)}{\overline{(p_n g_n)(\Phi(z))}} = 1 \qquad \text{for} \quad z \in L(x, \delta_0). \tag{3.47}$$

From (3.42) and (3.46) we deduce that

$$(1+\varepsilon_{\rm n})^{-n} \leqslant |g_n(z)| \leqslant (1-\varepsilon_n)^{-n} \tag{3.48}$$

for $z \in \overline{L(x, \delta_2)}$, $n \geqslant n_0$. Since for $\delta_0 > 0$ small the curve $\partial L(x, \delta)$ is close to a circle around x, there exists δ_1 with $0 < \delta_1 < \delta_2$ such that the two curves $\partial L(x, \delta_1)$ and $\partial L(x, \delta_2)$ can be separated by a circle around x. We know that g_n is analytic in $L(x, \delta_0)$ for n sufficiently large and we have the estimate (3.48). Approximating g_n by Taylor polynomials shows that there exist polynomials $h_n \in \mathscr{P}_{k_n}$ with $h_n(x) = 1$ and numbers $0 < \eta_1 < \eta_2 < 1$, $c_1 > 0$ such that

$$|(g_n - h_n)(z)| \leqslant c_1(1-\varepsilon_n)^{-n}\,(1-\eta_1)^{k_n} \tag{3.49}$$

for all $z \in \overline{L(x, \delta_1)}$ and $n \geqslant n_0$. Let r_n be the approximation error, i.e.,

$$h_n = g_n + r_n = g_n\left(1 + \frac{r_n}{g_n}\right). \tag{3.50}$$

Then from (3.48) and (3.49) we deduce the estimate

$$\left|\left(\frac{r_n}{g_n}\right)(z)\right| \leqslant c_1(1+\varepsilon_n)^{3n}\,(1-\eta_1)^{k_n} \tag{3.51}$$

for all $z \in \overline{L(x, \delta_1)}$ and $n \geqslant n_1$, which yields

$$\begin{aligned}\left|\frac{1+(r_n/g_n)(z)}{1+\overline{(r_n/g_n)(\Phi(z))}} - 1\right| &\leqslant 4c_1(1+\varepsilon^n)^{3n}\,(1-\eta_1)^{k_n}\\ &\leqslant 4c_1(1+\varepsilon_n)^{3n}\,(1-\eta_1)^{k_n} < (1-\eta_2)^{k_n}\end{aligned} \tag{3.52}$$

for all $z \in \overline{L(x, \delta_1)}$ and $n \geqslant n_2$. The second identity of (3.47) together with (3.52) gives the asymptotic estimate

$$\begin{aligned}\frac{(p_n h_n)(z)}{\overline{(p_n h_n)(\Phi(z))}} &= \frac{(p_n g_n)(z)}{\overline{(p_n g_n)(\Phi(z))}}\left(\frac{1+(r_n/g_n)(z)}{1+\overline{(r_n/g_n)(\Phi(z))}}\right)\\ &= 1 + \left(\frac{1+(r_n/g_n)(z)}{1+\overline{(r_n/g_n)(\Phi(z))}} - 1\right)\\ &= 1 + O((1-\eta_2)^{k_n}) \qquad \text{as} \quad n \to \infty\end{aligned} \tag{3.53}$$

uniformly for $z \in \overline{L(x, \delta_1)}$, which proves assertion (iii) of the lemma. Assertion (i) follows from (3.43).

For the proof of assertion (ii) we need some preparation. From (3.48) and (3.49) we deduce that

$$|h_n(z)| \leqslant c_2(1-\varepsilon_n)^{-n} \tag{3.54}$$

for $z \in \overline{L(x, \delta_1)}$, $n \geqslant n_2$, and $c_2 \in \mathbb{R}$. Thus, for any $\Theta > 1$ we have

$$|h_n(z)| \leqslant \Theta^n \tag{3.56}$$

for $z \in \overline{L(x, \delta_1)}$ and $n \geqslant n_2$. Using the fact that $h_n \in \mathscr{P}_{k_n}$ and Bernstein's lemma, we deduce that

$$|h_n(z)| \leqslant \Theta^{\mathrm{n}} \exp[k_n \, g_{\overline{\mathbb{C}} \setminus \overline{L(x, \delta_1)}}(z, \infty)] \tag{3.57}$$

for $z \in \mathbb{C}$ and $n \geqslant n_2$. With (3.43) this implies that

$$\limsup_{n \to \infty} |h_n(z)|^{1/n} \leqslant \Theta \tag{3.58}$$

locally uniformly for $z \in \mathbb{C}$. Since $\Theta > 1$ is arbitrary, (3.58) proves assertion (ii), which completes the proof of Lemma 3.3. ∎

LEMMA 3.4. *Let the domain D, the set F_1, $x \in F \setminus F_1$, and $\delta_0 > 0$ be defined as in Lemma* 3.3. *Set $J := F \cap \overline{L(x, \delta_0)}$, where $\delta_0 > 0$ is so small that J is an analytic arc, and let $p_n \in \mathscr{P}_n^*$, $n = 1, 2, \ldots$, be a sequence of polynomials and ν a positive measure such that*

(i) $(1/n)\, \nu_{p_n} \xrightarrow{*} \nu$ *as* $n \to \infty$,

(ii) $\lim_{n \to \infty} (p_n(z)/\overline{p_n(\Phi(z))}) = c_0$ *uniformly for* $z \in \overline{L(x,\delta_0)}$ *with* $|c_0| = 1$,

(iii) *all zeros of p_n on J are of even order, and*

(iv) $p(\nu; \cdot\,)$ *assumes its minimum $c_1 \in \mathbb{R}$ on J uniquely at the point x, i.e. $p(\nu; x) = c_1$ and $p(\nu; z) > c_1$ for all $z \in J \setminus \{x\}$.*

Further, let f be a continuous, complex-valued function defined on J, and $f(z) \neq 0$ for all $z \in J$. Then

$$\lim_{n \to \infty} \left| \int_J p_n(\zeta)\, f(\zeta)\, d\zeta \right|^{1/n} = e^{-c_1}. \tag{3.59}$$

Proof. By multiplying f by a constant $c_2 \in \mathbb{C}$, $|c_2| = 1$, and denoting the product again by f, we can assume that there exists $0 < \delta_1 < \delta_0$ such that

$$|\arg f(z)| \leqslant \frac{\pi}{12} \qquad \text{for all} \quad z \in J \cap \overline{L(x, \delta_1)}. \tag{3.60}$$

If $\delta_1 > 0$ is sufficiently small, and if we multiply $d\zeta$ by a constant $c_3 \in \mathbb{C}$, $|c_3| = 1$, we can further assume that

$$|\arg d\zeta| \leqslant \frac{\pi}{12} \qquad \text{on} \quad J \cap \overline{L(x, \delta_1)}, \tag{3.61}$$

where the orientation on J is assumed to be the same as in (3.59). From the assumptions (i) and (iv), and the principle of descent given in Lemma 2.1 and applied to $\log |p_n|^{-1}$, it follows rather immediately that

$$\limsup_{n \to \infty} \left| \int_J p_n(\zeta)\, f(\zeta)\, d\zeta \right|^{1/n} \leqslant e^{-c_1}. \tag{3.62}$$

Let the arc J be broken down in $J = J_1 \cup J_2$ with $J_1 := J \cap \overline{L(x, \delta_1)}$ and $J_2 := J \setminus J_1$. Because of assumption (iv) there exists $\delta_2 > 0$ such that

$$\mathrm{p}(v; z) \geqslant c_1 + \delta_2 \qquad \text{for all} \quad z \in J_2. \tag{3.63}$$

From (3.63), assumption (i), and the principle of descent (Lemma 2.1) we deduce as in (3.62) that

$$\limsup_{n \to \infty} \left| \int_{J_2} p_n(\zeta)\, f(\zeta)\, d\zeta \right|^{1/n} \leqslant e^{-\delta_2} e^{-c_1}. \tag{3.64}$$

Next, we estimate the integral over J_1, which is the more difficult part of the proof. Since $\Phi(z) = z$ for all $z \in J$, we have

$$\arg \left(\frac{p_n(z)}{\overline{p_n(\Phi(z))}} \right) = 2 \arg p_n(z) \qquad \text{for} \quad z \in J \setminus Z(p_n). \tag{3.65}$$

Multiplying p_n by a constant $c_4 \in \mathbb{C}$, $|c_4| = 1$, and denoting the product again by p_n, we deduce from the assumptions (ii) and (iii) that there exists $n_0 \in \mathbb{N}$ such that

$$|\arg p_n(z)| \leqslant \frac{\pi}{12} \qquad \text{for all} \quad z \in J,\ n \geqslant n_0. \tag{3.66}$$

At the points of $Z(p_n) \cap J$ the value of $\arg p_n$ is defined by continuous completion. From (3.60), (3.61), and (3.66) it follows that

$$\left| \int_{J_1} p_n(\zeta)\, f(\zeta)\, d\zeta \right| \geqslant \sqrt{\tfrac{1}{2}} \int_{J_1} |p_n(\zeta)|\, |f(\zeta)|\, d|\zeta| \qquad \text{for} \quad n \geqslant n_0. \tag{3.67}$$

since $|\arg(p_n f d\zeta)| \leqslant \pi/4$ on J_1. Comparing $p(v; \cdot) - p(v; x)$ with the function $-\|v\|\, g_{\overline{\mathbb{C}} \setminus J}(\cdot, \infty)$, it follows from Assumption (iv) that for any

$$0 < \varepsilon < \delta_2/5 \tag{3.68}$$

there exists $\delta_3 > 0$ such that

$$\overline{\mathbb{D}(x, \delta_3)} \subseteq L(x, \delta_1) \text{ and } p(v; z) \geqslant p(v; x) - \varepsilon \qquad \text{for all} \quad z \in \overline{\mathbb{D}(x,\delta_3)}. \tag{3.69}$$

From the assumptions (i) and (iv), the principle of descent (Lemma 2.1), and (3.69) it follows that there exists $n_1 \in \mathbb{N}$ such that

$$\|p_n\|_{\overline{\mathbb{D}(x, \delta_3)}} \leqslant e^{2n\varepsilon} e^{-nc_1} \qquad \text{for} \quad n \geqslant n_1, \tag{3.70}$$

where $\|\cdot\|_K$ denotes the sup-norm on the set K. By Bernstein's inequality about derivatives of polynomials we deduce from (3.70) that

$$\|p_n'\|_{\overline{\mathbb{D}(x, \delta_3)}} \leqslant \frac{n}{\delta_3} e^{2n\varepsilon} e^{-nc_1} \qquad \text{for} \quad n \geqslant n_1. \tag{3.71}$$

Let $|p_n|$ assume its maximum on $J \cap \overline{\mathbb{D}(x, \delta_3)}$ at the point $x_n \in J \cap \overline{\mathbb{D}(x, \delta_3)}$. It then follows from the assumptions (i) and (iv) and the Lower Envelope Theorem given in Lemma 2.2 that $x_n \to x$ and $|p_n(x_n)|^{1/n} \to e^{-c_1}$ as $n \to \infty$. Hence, there exists $n_2 \in \mathbb{N}$ such that

$$|p_n(x_n)| \geqslant e^{-n\varepsilon} e^{-nc_1} \qquad \text{for} \quad n \geqslant n_2. \tag{3.72}$$

We define

$$\eta_n := \frac{\delta_3}{2n} e^{-3n\varepsilon}. \tag{3.73}$$

If n_2 is chosen large enough, we have

$$\operatorname{dist}(x_n, \partial\mathbb{D}(x, \delta_3)) > \eta_n \qquad \text{for} \quad n \geqslant n_2. \tag{3.74}$$

From (3.71), (3.72), and (3.73) we deduce that

$$|p_n(z)| \geqslant |p_n(x_n)| - \eta_n \|p_n'\|_{\overline{\mathbb{D}(x, \delta_3)}} \geqslant \tfrac{1}{2} e^{-n\varepsilon} e^{-nc_1} \tag{3.75}$$

for all $z \in \overline{\mathbb{D}(x_n, \eta_n)}$. Since $\eta_n \to 0$ and $x_n \to x$ there exists $c_5 > 0$ such that

$$|f(z)| \geqslant c_5 \qquad \text{for all} \quad z \in J \cap \overline{\mathbb{D}(x_n, \eta_n)}. \tag{3.76}$$

From (3.75), (3.76), (3.73), and (3.74) it follows that

$$\begin{aligned} \int_{J_1} |p_n(\zeta)|\, |f(\zeta)|\, d|\zeta| &\geqslant \int_{J \cap \overline{\mathbb{D}(x_n, \eta_n)}} |p_n(\zeta)|\, |f(\zeta)|\, d|\zeta| \\ &\geqslant \frac{1}{2} e^{-n\varepsilon} e^{-nc_1} c_5 \frac{2}{2n} e^{-3n\varepsilon} \geqslant \frac{c_5}{2n} e^{-4n\varepsilon} e^{-nc_1}, \end{aligned} \tag{3.77}$$

and therefore with (3.67) that

$$\liminf_{n\to\infty}\left|\int_{J_1} p_n(\zeta)\, f(\zeta)\, d\zeta\right|^{1/n} \geqslant e^{-4\varepsilon} e^{-c_1}. \tag{3.79}$$

Using (3.64) and (3.68), we deduce from (3.79) that

$$\liminf_{n\to\infty}\left|\int_{J} p_n(\zeta)\, f(\zeta)\, d\zeta\right|^{1/n} \geqslant e^{-4\varepsilon} e^{-c_1}. \tag{3.80}$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, (3.80) together with (3.62) proves (3.59). ▮

4. PROOFS OF RESULTS IN SECTION 1

In the present section all the new results of Section 1 will be proved. We start with two preparatory lemmas. Then the first part of Theorem 1.8 will be proved. The proof of Theorem 1.7 follows, and after that the proofs of Theorems 1.1, 1.2, and 1.4, the Proof of the second part of Theorem 1.8, and finally the proof of Theorem 1.9 are presented.

LEMMA 4.1. *Let f be analytic near infinity and let C be a negatively oriented integration path such that f is analytic on C and in* $\mathrm{Ext}(C)$. *Further, let $(p_{mn}, q_{mn}) \in \mathcal{P}_m \times (\mathcal{P}_n \backslash \{0\})$, $m, n \in \mathbb{N}$, be a pair of Padé polynomials, i.e., let* (1.3) *hold true, and let $(\tilde{p}_{mn}, \tilde{q}_{mn})$ be a reduced pair such that $\tilde{p}_{mn}$ and $\tilde{q}_{mn}$ are coprime. The reversed denominator polynomial Q_n is defined by*

$$Q_n(z) := z^{\deg(\tilde{q}_{mn})}\, \tilde{q}_{mn}\left(\frac{1}{z}\right) \in \mathcal{P}_n^*. \tag{4.1}$$

The denominator polynomial Q_n satisfies the orthogonality relation

$$\oint_C \zeta^k Q_n(\zeta)\, \zeta^{m-n}\, d\zeta = 0 \qquad \textit{for} \quad k = 0, ..., n-1. \tag{4.2}$$

By the approximation error of the Padé approximant we have the representation

$$f(z) - [m/n](z) = \frac{1}{2\pi i} \frac{z^{n-m}}{(Q_n P)(z)} \oint_C \frac{(Q_n Pf)(\zeta)\zeta^{m-n}}{\zeta - z}\, d\zeta \tag{4.3}$$

for $z \in \mathrm{Ext}(C)$ and for any $P \in \mathcal{P}_n^*$. (The Padé approximant $[m/n]$ has been defined in (1.2).)

Remark. The pair of Padé polynomials $(\tilde{p}_{mn}, \tilde{q}_{mn})$ can be multiplied by a nonzero constant without violating (1.3). Therefore, the pair can be normalized such that $Q_n \in \mathscr{P}_n^*$, as has been assumed in (4.1). Note also that $0 \notin \mathscr{P}_n^*$.

Proof. Multiplying relation (1.3) by $z^{k+m-n+\tilde{n}}$, $k=0, ..., n-1$, $\tilde{n} := \deg(\tilde{q}_{mn})$, yields

$$z^k z^{m-n} Q_n(z) f(z) - z^{k+m-n+\tilde{n}} \tilde{p}_{mn}\left(\frac{1}{z}\right) = O(z^{k-n-1}) \qquad \text{as} \quad z \to \infty. \tag{4.4}$$

From Cauchy's Theorem it then follows that

$$\oint_C \zeta^{k+m-n+\tilde{n}} \tilde{p}_{mn}\left(\frac{1}{\zeta}\right) d\zeta = 0 \tag{4.5}$$

since $\deg(\tilde{p}_{mn}) \leqslant m-n+\tilde{n}$, and the same integral applied to the right-hand side is also zero since there is a zero at infinity of order at least two. This proves (4.2).

Multiplying relation (1.3) by $z^{m-n+\tilde{n}} P(z)$ with $P \in \mathscr{P}_n^*$ and using (4.1) yields

$$z^{m-n} P(z)\, Q_n(z)\, f(z) - P(z)\, z^{m-n+\tilde{n}} \tilde{p}_{mn}\left(\frac{1}{z}\right) = O(z^{-1}) \qquad \text{as} \quad z \to \infty. \tag{4.6}$$

Applying Cauchy's integral formula then gives

$$z^{m-n}(Q_n P)(z)[f(z) - [m/n](z)] = \frac{1}{2\pi i} \oint_C \frac{(Q_n Pf)(\zeta)\zeta^{m-n}}{\zeta - z}\, d\zeta \tag{4.7}$$

for $z \in \operatorname{Ext}(G)$, which proves (4.3). We have used the definition (1.2) of the Padé approximant. ∎

In Lemma 4.1 the integration path C was rather arbitrary. In the next lemma we show the construction of a "good" integration path for the integrals (4.2) and (4.3). This path will typically be used in the subsequent proofs. Let D be a symmetric domain as introduced in Definition 1.3, $F = \overline{\mathbb{C}} \backslash D$, C a negatively oriented integration path with $F \subseteq \operatorname{Int}(C)$, and $F_1 \subseteq F$ a compact set with $F_0 \subseteq F_1$ and $\operatorname{cap}(F_1) = 0$. Further, let $V \subseteq \mathbb{C}$ be an open set with $F_1 \subseteq V$. Then there exists a chain of smooth curves C_1 *in* $V \backslash F_1$ that is homologous to C in $\overline{\mathbb{C}} \backslash F_1$. Analoguously to (1.24), $\{J_j\}_{j \in I}$ now denotes the family of open arcs that form $F \backslash F_1$. It has been assumed in Definition 1.3 that $F \backslash F_0 \neq \varnothing$. If V is small enough, then the same is true for C_1, and we have $F \backslash \operatorname{Int}(C_1) \neq \varnothing$. Like $F \backslash F_1$, the set $F \backslash \operatorname{Int}(C_1)$ consists of analytic arcs, but now these arcs are closed. They are denoted by Γ_j, $j \in I_1$; i.e.,

$$\bigcup_{j \in I_1} \Gamma_j = F \setminus \mathrm{Int}(C_1). \tag{4.8}$$

Each Γ_j, $j \in I_1$, is a subarc of some $J_{j'}$, $j' \in I$.

LEMMA 4.2. *Let Γ_{j+} and Γ_{j-} denote the two banks of the arc Γ_j, $j \in I_1$, each with opposite orientations. Then the orientation of each pair $(\Gamma_{j+}, \Gamma_{j-})$ can be chosen in such a way that the chain of arcs and curves*

$$C_0 := C_1 + \sum_{j \in I_1} (\Gamma_{j+} + \Gamma_{j-}) \tag{4.9}$$

is closed and homologous to C in $\overline{\mathbb{C}} \setminus F_1$, and there exists a chain of curves $\tilde{C}_0$ arbitrarily close to C_0, which is fully contained in D, and $\tilde{C}_0$ is homologous to C in D.

Remark. The chain C_0 will be the standard integration path in the analysis that follows. Note that the main part of C_0 is contained in $F \setminus F_1$; only C_1 lays, except for isolated points, outside of F_1, but since C_1 is contained in the open set V it can be chosen arbitrarily near to F_1. This last property will be essential for proving integral estimates. The chain of curves C_1 can always be assumed to consist of only finitely many smooth curves.

Proof. Let V_1 be an open set with $\overline{V_1} \subseteq V$, $F_1 \subseteq V_1$, and ∂V_1 be smooth and consist of finitely many closed curves. Let C_1 be ∂V_1 with an appropriate orientation, and consider the level lines

$$C_1' := \{z \in \mathbb{C} \setminus V_1 \mid g_D(z, \infty) = r\}, \qquad r > 0. \tag{4.10}$$

The set C_r' consists of finitely many analytic arcs. For $r \to 0$ these arcs tend to the arcs Γ_j, $j \in I_1$, from both sides of $F \setminus V_1$, which is a consequence of Lemma 2.9. For $r > 0$ sufficiently small let the domain V_2 be defined by

$$V_2 := \{z \in \overline{\mathbb{C}} \setminus V_1 \mid g_D(z, \infty) > r\}. \tag{4.11}$$

The boundary ∂V_2 with negative orientation is denoted by $\tilde{C}_0$; it is homologous to C. Since I_1 is finite, all arcs in $C_r{}'$ are homotopic to $\{\Gamma_{j+}, \Gamma_{j-}\}_{j \in I_1}$ for $r > 0$ sufficiently small, and (4.10) together with Lemma 2.9 defines for $r_0 \geqslant r > 0$ a homotopic variation of C'_{r_0}. This shows that C_0 is also homologous to C. ∎

Proof of Theorem 1.8 (First Part). We prove that the first limit in (1.28) holds true under the assumptions made in Theorem 1.7. The full theorem, i.e. the second limit in (1.28) under the assumptions of Theorem 1.7 and both limits in (1.28) under the assumptions made in Theorem 1.1, will be proved later. In order to simplify the notation, we assume that in the sequence of indices $\{(m_j, n_j)\}$ each component appears at most once, which implies that we can write $n := n_j$ and $m = m(n) := m_j$, $n \in N \subseteq \mathbb{N}$. Since the sequence $\{(m, n)\}_{n \in N}$ has to satisfy (1.6), it follows that

$$\frac{m_j - n_j}{n_j} = \frac{m - n}{n} \to 0 \qquad \text{as} \quad n \to \infty. \tag{4.12}$$

With the definition of the reversed denominator polynomials Q_n in (4.1) we define

$$\pi_n := \frac{1}{n} \pi_{m_j n_j} = \frac{1}{n} \nu_{Q_n}, \tag{4.13}$$

and prove

$$\pi_n \overset{*}{\to} \omega \qquad \text{as} \quad n \to \infty, \tag{4.14}$$

where $\omega = \omega_F$ is the equilibrium distribution on F. The convergence (4.14) will be proved indirectly. We assume that there exists an infinite subsequence $N \subseteq \mathbb{N}$ such that

$$\pi_n \overset{*}{\to} \mu \neq \omega \qquad \text{as} \quad n \to \infty, \qquad n \in N. \tag{4.15}$$

It is immediate that $\mu \geqslant 0$ and $\|\mu\| \leqslant 1$. Thus, the assumptions of Lemma 3.1 are satisfied.

As already mentioned earlier, Lemma 3.1 will be the main tool of the proof. From the assumptions of Theorem 1.7 it follows that there exists a compact set $\tilde{F}_1 \subseteq F$ with $\operatorname{cap}(\tilde{F}_1) = 0$ such that the jump functions g_j, $j \in I$, defined in (1.24) for each arc J_j, $j \in I$, are continuous and different from zero on $J_j \backslash \tilde{F}_1$. We define

$$F_1 := F_0 \cup \{0\} \cup \tilde{F}_1. \tag{4.16}$$

Then $\operatorname{cap}(F_1) = 0$. From Lemma 3.1 it follows that there exists $\delta_0 > 0$, $x \in F \backslash F_1$ with $\overline{L(x, \delta_0)} \subseteq U \backslash F_1$, and a measure ν with $\nu \geqslant 0$ and $|\nu| < 1$ such that the assertions (i) through (iv) in Lemma 3.1 hold true.

With the help of the measure ν a sequence of polynomials $P_n \in \mathscr{P}_n{}^*$, $n \in N$ is defined in several steps. We define

$$\nu_{1,n} := \Phi(\pi_n|_{L(x, \delta_0)}), \qquad n \in N, \tag{4.17}$$

and assume that $P_{1,n} \in \mathscr{P}_n^*$ is chosen such that $p(\nu_{1,n}; \cdot) = -(1/n) \log |P_{1,n}|$. Without loss of generality we can assume that $\mu(\partial L(x, \delta_0)) = 0$ since otherwise we can choose a smaller $\delta_0 > 0$, for which this assumption as well as Lemma 3.1 is true. It then follows from (4.15) that there exists a positive measure ν_1 such that

$$\nu_{1,n} \xrightarrow{*} \nu_1 \qquad \text{as} \quad n \to \infty, \, n \in N. \tag{4.18}$$

We have

$$\frac{1}{n} \deg(P_{1,n}) \to |\nu_1| \qquad \text{as} \quad n \to \infty, \, n \in N, \tag{4.19}$$

and because of (4.15), (4.17), and assertion (iv) in Lemma 3.1, it follows that

$$\nu_2 := \nu - \nu_1 \geqslant 0 \qquad \text{and} \qquad \operatorname{supp}(\nu_2) \cap L(\mathrm{x}, \delta_0) = \varnothing. \tag{4.20}$$

In the next step for each $n \in N$ we select $[n \, \|\nu_2\|]$ points from $\operatorname{supp}(\nu_2)$ in such a way that the sequence of polynomials $P_{2,n} \in \mathscr{P}_n^*$ satisfies

$$\frac{1}{n} \nu_{P_{2,n}} \xrightarrow{*} \nu_2 \qquad \text{as} \quad n \to \infty, \, n \in N. \tag{4.21}$$

We have

$$\frac{1}{n} \deg(P_{2,n}) \to \|\nu_2\| \qquad \text{as} \quad n \to \infty, \, n \in N. \tag{4.22}$$

Define

$$\nu_{3,n} := \frac{m-n}{n} \delta_0, \qquad P_{3,n}(z) := z^{m-n}. \tag{4.23}$$

Actually, $P_{3,n}$ is a polynomial only if $m \geqslant n$, otherwise $P_{3,n}$ is a rational function. However, the functions $P_{3,n}$ are very special, and it seems that all conclusions are rather immediate and clear without giving to this aspect any special consideration. We have

$$\nu_{3,n} \xrightarrow{*} 0 \qquad \text{as} \quad n \to \infty, \, n \in N, \tag{4.24}$$

and

$$\limsup_{n \to \infty,\, n \in N} |P_{3,n}(z)|^{1/n} \leqslant 1 \tag{4.25}$$

locally uniformly in $\mathbb{C} \backslash \{0\}$.

Set

$$c_1 := p(\nu + \mu; x). \tag{4.26}$$

From assertion (ii) of Lemma 3.1 we deduce that there exists $\varepsilon_1 > 0$ such that

$$p(\nu + \mu; z) \geqslant c_1 + \varepsilon_1 \qquad \text{for} \quad z \in F \backslash L(x, \delta_0), \tag{4.27}$$

and assertion (i) of Lemma 3.1 implies that there exists an open set $V \subseteq \mathbb{C}$ with $F_1 \subseteq V$,

$$\bar{V} \cap \overline{L(x, \delta_0)} = \varnothing, \tag{4.28}$$

and

$$p(\nu + \mu; z) \geqslant c_1 + \varepsilon_1 \qquad \text{for} \quad z \in \bar{V}. \tag{4.29}$$

From the principle of descent (Lemma 2.1) and the limits (4.15), (4.18), (4.21), and (4.24) it follows that

$$\limsup_{n \to \infty,\, n \in N} |(Q_n \; P_{1,n} P_{2,n} P_{3,n})(z)|^{1/n} \leqslant \exp[-p(\mu + \nu; z)] \leqslant e^{-c_1 - \varepsilon_1} \tag{4.30}$$

for all $z \in (\bar{V} \cup F) \backslash L(x, \delta_0)$, and the overall asymptotic inequality, i.e. the inequality between the limsup on the leftside of (4.30) and $e^{-c_1 - \varepsilon_1}$ on the right-hand side, holds uniformly on $(\bar{V} \cup F) \backslash L(x, \delta_0)$.

Let $x_n \in F$, $n \in N$, be a sequence with $x_n \to x$ as $n \to \infty$, $n \in N$, and $P_{1,n}(x_n) \neq 0$, $Q_n(x_n) \neq 0$. Define

$$\tilde{P}_n := \frac{P_{1,n} P_{2,n} P_{3,n}}{\arg(Q_n P_{1,n} P_{2,n} P_{3,n})(x_n)}, \qquad n \in N. \tag{4.31}$$

Because of (4.17) and (4.16), we have

$$Z(Q_n \tilde{P}_n)|_{L(x, \delta_0)} = \Phi(Z(Q_n \tilde{P}_n)|_{L(x, \delta_0)}), \tag{4.32}$$

and from (4.32), assertion (iii) of Lemma 3.1, and the limits (4.15), (4.18), (4.21), and (4,24) we deduce that

$$\lim_{n\to\infty,\, n\in N} \frac{1}{n} \log \left| \frac{(Q_n \widetilde{P}_n)(z)}{(Q_n \widetilde{P}_n)(\Phi(z))} \right| = p(\mu + \nu; z) - p(\mu + \nu, \Phi(z)) = 0 \tag{4.33}$$

locally uniformly for $z \in L(x, \delta_0)$. From Lemma 3.3, (4.31), (4.32), and (4.33) it follows that there exist $P_{4,n} \in \mathscr{P}_n^*$, $n \in \mathbb{N}$, with

$$\frac{1}{n} \deg(P_{4,n}) \to 0 \qquad \text{as} \quad n \to \infty,\ n \in N, \tag{4.34}$$

$$\limsup_{n\to\infty,\, n\in N} |P_{4,n}(z)|^{1/n} \leqslant 1 \tag{4.35}$$

locally uniformly for $z \in \mathbb{C}$, and

$$\lim_{n\to\infty,\, n\in N} \frac{(Q_n P_n)(z)\, z^{m-n}}{(Q_n P_n)(\Phi(z))\, \Phi(z)^{m-n}} = 1 \tag{4.36}$$

locally uniformly for $z \in L(x, \delta_0)$, where

$$P_n := \widetilde{P}_n P_{4,n} / P_{3,n} \tag{4.37}$$

and the normalization used in (4.31) as well as in (4.23) has been used to secure the assumption (3.38) of Lemma 3.3. Note that the quotient in (4.36) is analytic in $L(x, \delta_0)$. From (4.19), (4.22), and (4.34) if follows that

$$\lim_{n\to\infty,\, n\in N} \frac{1}{n} \deg(P_n) = \|\nu\| < 1. \tag{4.38}$$

The expression $(Q_n P_n f)(z) z^{m-n}$ is analytic in $D \backslash F_1 = D \backslash \{0\}$ and possesses a continuous continuation from D to $\partial D \backslash F_1$ with different boundary values from both sides of $F \backslash F_1$. From Lemma 4.2 and the definition of the jump functions g_j, $j \in I$, in (1.25), it follows that

$$\begin{aligned} \oint_C (P_n Q_n f)(\zeta) \zeta^{m-n}\, d\zeta &= \oint_{C_0} (P_n Q_n f)(\zeta) \zeta^{m-n}\, d\zeta \\ &= \oint_{C_1} (P_n Q_n f)(\zeta) \zeta^{m-n}\, d\zeta \\ &\quad + \sum_{j \in I_1} \int_{\Gamma_j} (P_n Q_n g_j)(\zeta) \zeta^{m-n}\, d\zeta. \end{aligned} \tag{4.39}$$

Because of (4.8) and the remarks made after (4.8) there exists $j' \in I$ such that $x \in J_{j'}$ and correspondingly $j \in I_1$ with

$$F \cap L(x, \delta_0) = J_{j'} \cap L(x, \delta_0) = \Gamma_{\mathrm{j}} \cap L(x, \delta_0). \tag{4.40}$$

Let C_2 denote the integration path C_0 without the two subarcs $\Gamma_{j\pm} \cap L(x, \delta_0)$. (For a definition of Γ_{j+} and Γ_{j-} see Lemma 4.2). Since $\mathrm{Length}(C_2) < \infty$ it follows from (4.30) that

$$\limsup_{n \to \infty,\, n \in N} \left| \oint_{C_2} (P_n Q_n f)(\zeta) \zeta^{m-n} \, d\zeta \right|^{1/n} \leqslant e^{-c_1 - \varepsilon_1}. \tag{4.41}$$

On the other hand, the limits (4.15), (4.18), (4.21), and (4.24); the definition of P_n in (4.37); and the definition of F_1 in (4.16) shows that Lemma 3.4 is applicable, and we deduce from (3.59) in this lemma, from (4.26), and from assertion (ii) of Lemma 3.1 that

$$\lim_{n \to \infty,\, n \in N} \left| \int_{F \cap L(x, \delta_0)} (P_n Q_n g_j)(\zeta) \zeta^{m-n} \, d\zeta \right|^{1/n} = e^{-c_1}. \tag{4.42}$$

From (4.39), (4.41), and (4.42) it follows that

$$\lim_{n \to \infty,\, n \in N} \left| \oint_{C} (P_n Q_n f)(\zeta) \zeta^{m-n} \, d\zeta \right|^{1/n} = e^{-c_1}. \tag{4.43}$$

From (4.38) we immediately deduce that $\deg(P_n) < n$ for $n \in N$ sufficiently large, and from the orthogonality relation (4.2) of Lemma 4.1 we then know that

$$\oint_{C} (P_n Q_n f)(\zeta) \zeta^{m-n} \, d(\zeta) = 0 \tag{4.44}$$

for $n \in N$ sufficiently large. Since (4.43) and (4.44) contradict each other, assumption (4.15) must be false, and therefore we have proved that (4.14) holds true, which completes the first part of the proof of Theorem 1.8. ▮

The second part of the proof of Theorem 1.8 will be given below, after the proofs of Theorems 1.7, 1.1, 1.2, and 1.4 have been performed.

Proof of Theorem 1.7. The proof is divided into two parts: in the first part the limit (1.9) is proved and in the second the limit (1.10) is proved. In both cases the error formula (4.3) will play a basic role. In order to prove limit (1.9) it is sufficient to derive an upper estimate of the integral in this formula, while the second part of the proof demands a more careful investigation of the asymptotic behavior of this integral. For this investigation Lemma 3.2 is needed.

As in the proof of Theorem 1.8 (First Part), we assume for the simplification of notation that in the sequence of indices $\{(m_j, n_j)\}_{j\in\mathbb{N}}$ each n_j appears at most once. Thus, we can write n instead of n_j and $m=m(n)$ instead of m_j. By $N\subseteq\mathbb{N}$ we denote the subsequence of all numbers n. Since in Theorem 1.7 it has been assumed that the sequence $\{(m, n)\}_{n\in N}$ satisfies (1.6), it follows that the limit (4.12) holds true.

Proof of Limit (1.9). Define

$$H_n(z) := \frac{1}{2\pi i}\oint_C \frac{Q_n^2(\zeta)\, f(\zeta)\, \zeta^{m-n}}{\zeta - z}\, d\zeta, \qquad z\in \operatorname{Ext}(C), \tag{4.45}$$

where C is a negatively oriented integration path around infinity such that f is analytic in $\operatorname{Ext}(C)$ and on C. The path C can be varied in the domain D without changing the value of $H_n(z)$. The denominator polynomial $Q_n\in\mathscr{P}_n^*$ has been introduced in Lemma 4.1. With the abbreviation (4.45) the error function (4.3) can be written as

$$(f-[m/n])(z) = \frac{z^{m-n}}{Q_n^2(z)}\, H_n(z), \qquad z\in\operatorname{Ext}(C). \tag{4.46}$$

Using orthogonality (4.2), we deduce that

$$\frac{1}{2\pi i Q_n(z)}\oint_C \frac{(Q_n^2 f)(\zeta)\zeta^{m-n}}{\zeta - z}\, d\zeta - \frac{1}{2\pi i P(z)}\oint_C \frac{(PQ_n f)(\zeta)\zeta^{m-n}}{\zeta - z}\, d\zeta$$
$$= \frac{1}{2\pi i (PQ_n)(z)}\oint_C \frac{Q_n(\zeta)\, P(z) - Q_n(z)\, P(\zeta)}{\zeta - z}\,(Q_n f)(\zeta)\zeta^{m-n}\, d\zeta = 0, \tag{4.47}$$

which implies that

$$H_n(z) = \frac{Q_n(z)}{2\pi i P(z)}\oint_C \frac{(PQ_n f)(\zeta)\, \zeta^{m-n}}{\zeta - z}\, d\zeta \tag{4.48}$$

for any $P\in\mathscr{P}_n^*$ and $z\in\operatorname{Ext}(C)$.

From Lemma 2.9 we know that $g_D(z,\infty)=0$ for all $z\in F\backslash F_0$, where F_0 is the set introduced in (1.24) in the definition of the symmetric domain D. In this definition it has been assumed that $\operatorname{cap}(F\backslash F_0)>0$. It follows from representation (2.21) of the Green function in Lemma 2.6 and from Lemma 2.9 that there exists a constant $c_0\in\mathbb{R}$ such that

$$p(\omega; z) = -c_0 \qquad \text{for all} \quad z\in F\backslash F_0 \tag{4.49}$$

and $\omega = \omega_F$ denotes the equilibrium distribution on F. In the special case of $F \subseteq \mathbb{D}$ we have $c_0 = \log \operatorname{cap}(F)$ (cf. Remark 3 after Lemma 2.6). Our immediate aim is to show that

$$\limsup_{n\to\infty,\, n\in N} \frac{1}{m+n} \log |H_n(z)| \leqslant c_0 \tag{4.50}$$

locally uniformly for $z \in D$. Indeed, let $V \subseteq D \backslash \{0\}$ be a compact set and $\varepsilon > 0$; further, let the set F_1 be defined as in (4.16) in the proof of Theorem 1.8 (first part), i.e., $F_1 = F_0 \cup \{0\} \cup \tilde{F}_1$, where $\tilde{F}_1 \subseteq F$ is a compact set with $\operatorname{cap}(\tilde{F}_1) = 0$ such that the jump functions g_j, $j \in I$, defined in (1.25) for each arc J_j of $F \backslash F_0$, are continuous and different from zero on $J_j \backslash \tilde{F}_1$. From Lemma 2.11 we know that there exists a probability measure μ_1 with $\operatorname{supp}(\mu_1) \subseteq F$ such that (2.35) holds true. This implies that there exists $1 > \delta_1 > 0$ such that

$$|p(\delta_1(\omega - \mu_1); z)| \leqslant \varepsilon \qquad \text{for all} \quad z \in \mathrm{V}, \tag{4.51}$$

and

$$p(\delta_1 \mu_1 + (1-\delta_1)\omega; z) \geqslant -c_0 - \varepsilon \qquad \text{for all} \quad z \in F. \tag{4.52}$$

The last inequality is possible because of (4.49). Since $p(\delta_1 \mu_1 + (1-\delta_1)\,\omega; z) = \infty$ for all $z \in F_1$, there exists an open set V_1 with $F_1 \subseteq V_1$, $\overline{V_1} \cap V = \emptyset$, and

$$p(\delta_1 \mu_1 + (2-\delta_1)\omega; z) \geqslant -2c_0 - \varepsilon \qquad \text{for all} \quad z \in V_1. \tag{4.53}$$

In Lemma 4.2 it has been shown that there exists an integration path C_0 as given in (4.9) for the integral in (4.45), which is completely contained in $F \cup V_1$. Let the polynomials $P_n \in \mathscr{P}_m^*$, $m = m(n)$, $n \in N$, be chosen such that

$$\frac{1}{n} \nu_{P_n} \overset{*}{\to} \delta_1 \mu_1 + (1-\delta_1)\omega \qquad \text{as} \quad n \to \infty, \qquad n \in N. \tag{4.54}$$

Because of (4.12) such polynomials can always be selected. It follows from (4.13) and the limit (4.14) which was proved under the assumptions of Theorem 1.7 in the proof of Theorem 1.8 (first part) that

$$\frac{1}{n} \nu_{Q_n} \overset{*}{\to} \omega \qquad \text{as} \quad n \to \infty, \qquad n \in N. \tag{4.55}$$

The two limits (4.54) and (4.55) together with the principle of descent (Lemma 2.1), the estimates (4.52) and (4.53), and the limit (4.12) yield that

$$\limsup_{n\to\infty,\, n\in N} \frac{1}{m+n} \log |(Q_n P_n)(z)| \leqslant c_0 + \frac{\varepsilon}{2} \tag{4.56}$$

uniformly for $z \in C_0$. (In our notation we do not differentiate between C_0 and its impression.) As length$(C_0) < \infty$, it follows from (4.56) that

$$\limsup_{n\to\infty,\, n\in N} \frac{1}{m+n} \log \left| \frac{1}{2\pi i} \oint_{C_0} \frac{(Q_n P_n f)(\zeta)\zeta^{m-n}}{\zeta - z}\, d\zeta \right| \leqslant c_0 + \frac{\varepsilon}{2} \tag{4.57}$$

uniformly for $z \in V$.

From the estimate (4.51) and the two limits (4.54) and (4.55) it follows that

$$\lim_{n\to\infty,\, n\in N} \left| \frac{1}{n} \log \left| \frac{Q_n(z)}{P_n(z)} \right| \right| \leqslant \varepsilon \tag{4.58}$$

uniformly for $z \in V$. With the estimates (4.57) and (4.58) we deduce fom (4.48) and (4.12) that

$$\limsup_{n\to\infty,\, n\in N} \frac{1}{m+n} \log |H_n(z)| \leqslant c_0 + \varepsilon \tag{4.59}$$

uniformly for $z \in V$. Since V and $\varepsilon > 0$ were arbitrary, the asymptotic estimate (4.50) is proved.

We are now prepared to finish the proof of limit (1.9). Let $V \subseteq D \backslash \{\infty\}$ be an arbitrary compact set and $\tilde{\varepsilon} > 0$. Define

$$\tilde{E}_n := \{ z \in V \mid |f - [m/n])(z)| > (G(z) + \tilde{\varepsilon})^{m+n} \}. \tag{4.60}$$

There exists $\varepsilon > 0$ such that

$$\log \left| 1 + \frac{\tilde{\varepsilon}}{G(z)} \right| \geqslant 2\varepsilon \qquad \text{for all} \quad z \in V. \tag{4.61}$$

The function G has been defined in (1.8). Note that under the assumption that D is a symmetric domain, it follows from Definition 1.3 that $\operatorname{cap}(F \backslash F_0) > 0$, and therefore also that $\operatorname{cap}(F \backslash F_1) > 0$, and consequently $G(z) > 0$ for all $z \in D \backslash \{\infty\}$. From (1.8) and (4.49) we deduce that

$$\log \mathrm{G}(z) = -g_D(z, \infty) = p(\omega; z) + c_0. \tag{4.62}$$

From the estimate (4.50) and error formula (4.46) it follows that

$$\frac{1}{m+n}\log\frac{|(f-[m/n])(z)|}{G(z)^{m+n}}$$
$$\leqslant -\frac{2}{m+n}\log|Q_n(z)|-\log G(z)+\frac{|m-n|}{m+n}\log|z|+c_0+\varepsilon \tag{4.63}$$

for all $z\in V$ and n sufficiently large. The sequence of functions

$$f_n(z):=-\frac{2}{m+n}\log|Q_n(z)|-\log G(z)+c_0=p\left(\frac{2}{m+n}\,v_{Q_n}-\omega;z\right), \tag{4.64}$$

$n\in N$, will be investigated in more detail. The last equality in (4.64) follows from (4.62). Because of the limits (4.55) and (4.12) it follows from Corollary 2.15 that $f_n\to 0$ in capacity in D as $n\to\infty$, $n\in N$. From (4.12) it further follows that $((m-n)/(m+n))\log|z|\to 0$ for all $z\in\mathbb{C}\backslash\{0\}$. From Lemma 2.16 we know that the sum of two sequences converges in capacity if the two sequences converge in capacity separately. Hence, it follows that the right-hand side of (4.63) converges to 0 in capacity in D. As a consequence it follows that the sets

$$E_n:=\left\{z\in V\;\middle|\;-\frac{2}{m+n}\log|Q_n(z)|-\log \mathrm{G}(z)+\frac{m-n}{m+n}\log|z|+c_0>\varepsilon\right\} \tag{4.65}$$

satisfy

$$\operatorname{cap}(E_n)\to 0\qquad\text{as}\quad n\to\infty,\quad n\in N. \tag{4.66}$$

From (4.61), (4.63), and (4.65) we conclude that

$$\frac{1}{m+n}\log\frac{|(f-[m/n])(z)|}{G(z)^{m+n}}$$
$$\leqslant\frac{2}{m+n}\log|Q_n(z)|-\log G(z)+\frac{m-n}{m+n}\log|z|+c_0+\varepsilon$$
$$\leqslant 2\varepsilon\leqslant\log\left|1+\frac{\tilde{\varepsilon}}{G(z)}\right| \tag{4.67}$$

holds for all $z \in V \backslash E_n$ and $n \in N$ sufficiently large. The definitions (4.60) and (4.65) together with (4.61) and (4.67) imply that $\tilde{E}_n \subseteq E_n$ for $n \in N$ sufficiently large. Hence, $\operatorname{cap}(\tilde{E}_n) \to 0$ as $n \to \infty$, $n \in N$, follows from (4.66), which completes the proof of limit (1.9).

Proof of Limit (1.10). The most difficult part of the proof is to show that $(1/(m+n)) \log |H_n(z)|$ converges to c_0 in capacity in the domain D. For this purpose it is no longer sufficient to know only the asymptotic upper estimate for H_n as in (4.50).

Assume that D_1 is a subdomain of D with $\overline{D_1} \subseteq D$ and $\infty \in D_1$. Since the functions $-(1/(m+n)) \log |H_n(z)|$ are superharmonic in D, we can, as in (2.40), introduce the decomposition

$$\frac{1}{m+n} \log |H_n(z)| = \hat{h}(z) - g_{D_1}(\nu_n; z), \tag{4.68}$$

where $\hat{h}_n$ is harmonic in D_1 and $g_{D_1}(\nu_n; \cdot)$ is the Green potential of the non-negative measure ν_n in the domain D_1. We have that $\operatorname{supp}(\nu_n) \subseteq \overline{D_1}$ and $(m+n)\, \nu_n$ is the counting measure of the zeros of the function H_n in the domain D_1.

For an indirect proof we assume that the functions $(1/(m+n)) \log |H_n(z)|$ do not converge in capacity to the constant c_0 in the domain D. Then because of Lemma 2.14 and the asymptotic estimate (4.50) at least one of the following two assertions holds true: we have

$$\limsup_{n \to \infty,\, n \in N} \|\nu_n\| > 0 \tag{4.69}$$

or

$$\liminf_{n \to \infty,\, n \in N} \hat{h}_n(z) < c_0 \qquad \text{for} \quad z \in D_1. \tag{4.70}$$

In either case it follows that there exists an infinite subsequence of N, which we continue to denote by N, and a constant

$$c_1 < c_0 \tag{4.71}$$

such that there exists a closed set $V_2 \subseteq D_1$ with $\infty \in \operatorname{Int}(V_2)$ and ∂V_2 a smooth curve endowed with a negative orientation, the curve is denoted C_2, and we have

$$\limsup_{n \to \infty,\, n \in N} \frac{1}{m+n} \log |H_n(z)| \leqslant c_1 \qquad \text{uniformly for all} \quad z \in C_2. \tag{4.72}$$

Let ε_2 satisfy $0<\varepsilon_2<(c_0-c_1)/3$. From the special normalization used in (2.3) in the definition of the logarithmic potential it follows that there exist $0<\varepsilon_1<(c_0-c_1)/|3c_0|$ such that

$$p(v; z) \geqslant -\varepsilon_2 \qquad \text{for all} \quad z \in C_2 \tag{4.73}$$

and all positive measures v on $\mathbb{C}$ with $\|v\| \leqslant \varepsilon_1$. Let the sequence of polynomials $\tilde{p}_n \in \mathscr{P}^*_{[n\,\|v\|]}$ be chosen in such a way that

$$\frac{2}{m+n}\, v_{\tilde{p}_n} \overset{*}{\to} v \qquad \text{as} \quad n \to \infty,\ n \in N. \tag{4.74}$$

The choice is possible because of the limit (4.12). From (4.74), (4.73) and the principle of descent (Lemma 2.1) it follows that

$$\limsup_{n\to\infty,\, n\in N} \frac{1}{m+n} \log |\tilde{p}_n^2(z)| \leqslant \varepsilon_2 \qquad \text{uniformly for all} \quad z \in C_2. \tag{4.75}$$

From the limits (4.72) and (4.75) we deduce that

$$\frac{1}{m+n} \log \left| \oint_{C_2} (\tilde{p}_n^2 H_n)(z)\, dz \right| \leqslant c_1 + \varepsilon_2 < (1 + \|v\|)\ c_0 - \varepsilon_2 \tag{4.76}$$

for all $n \in N$, $n \geqslant n_2(\{\tilde{p}_n\})$. The last inequality in (4.76) is a consequence of (4.71) and the three inequalities $\varepsilon_2 < (c_0 - c_1)/3$, $\varepsilon_1 < (c_0 - c_1)/|3c_0|$, and $\|v\| \leqslant \varepsilon_1$.

We choose v to be a measure as described in Lemma 3.2; the number $\varepsilon > 0$ and F_1 in Lemma 3.2 are taken to be ε_1 and F_1 as defined in (4.16). By using the assertions proved in Lemma 3.2, we shall show that the first inequality in (4.76) cannot be true, which proves the convergence in capacity of $(1/(m+n)) \log |H_n(z)|$ to c_0 in D.

Since the point $x \in F\backslash F_1$ and the number $\delta_0 > 0$ are assumed to be the same as in Lemma 3.2, the assertions (i) through (iv) of Lemma 3.2 hold true. In addition we assume that $\delta_0 > 0$ is chosen so small that

$$\overline{L(x, \delta_0)} \cap \overline{V_2} = \varnothing. \tag{4.77}$$

Let polynomials $P_{n,3} \in \mathscr{P}^*_n$, $n \in N$, be chosen in such a way that the weak limit (4.74) holds true with $\tilde{P}_n$ substituted by $P_{n,3}$, and v is now the specially chosen measure in accordance with Lemma 3.2. For each $n \in N$ the polynomials $P_{n,1}, P_{n,2} \in \mathcal{P}_n^*$ are defined by

$$\begin{aligned} v_{P_{n,1}} &= v_{Q_n}|_{\mathbb{C} \setminus L(x, \delta_0)}, \\ v_{P_{n,2}} &= \Phi(v_{Q_n}|_{L(x, \delta_0)}), \end{aligned} \tag{4.78}$$

i.e., $P_{n,1}$ has the same zeros as Q_n outside of $L(x, \delta_0)$ and no zeros on $L(x, \delta_0)$, while $P_{n,2}$ has no zeros outside of $L(x, \delta_0)$; on $L(x, \delta_0)$ the zeros of $P_{n,2}$ are the image of zeros of Q_n under the reflection mapping Φ, which has been introduced in (2.29) of Definition 2.1. From limit (4.55) together with the Φ-symmetry of ω, (4.75), and (4.76), we deduce that

$$\lim_{n \to \infty, n \in N} \frac{1}{m+n} \log \left| \frac{Q_n(z)}{(P_{n,1\,P}{}^{n,2})(z)} \right| = 0 \tag{4.79}$$

uniformly for all $z \in \overline{V_2}$, which implies with (4.48) and (4.76) that for $n_3 \in \mathbb{N}$ sufficiently large we have

$$\begin{aligned} &\frac{1}{m+n} \log \left| \frac{1}{2\pi i} \oint_{C_2} P_{n,3}^2(z) \oint_C \frac{(P_{n,1} P_{n,2} Q_n f(\zeta) \zeta^{m-n}}{\zeta - z} \, d\zeta \, dz \right| \\ &\quad \leqslant (1 + \|v\|)\, c_0 - \varepsilon_2 \end{aligned} \tag{4.80}$$

for all $n \in N$, $n \geqslant n_3$. In order for the function H_n in (4.76) to be defined, we had to assume that $V_2 \subseteq [\mathrm{Ext}](C)$, or equivalently $C \subseteq \mathrm{Int}(C_2)$. Interchanging integration in (4.80) yields

$$\frac{1}{m+n} \log \left| \oint (P_n Q_n f)(\zeta) \zeta^{m-n} \, d\zeta \right| \leqslant (1 + \|v\|)\, c_0 - \varepsilon_2 \tag{4.81}$$

for all $n \in N$, $n \geqslant n_3$. In (4.81) the polynomial P_n is defined as

$$P_n := P_{n,3}^2 P_{n,2} P_{n,1}. \tag{4.82}$$

By the techniques of estimation of line integrals as applied after (4.26) in the proof of Theorem 1.8 (first part), we shall next prove an inequality that contradicts (4.81).

From assertion (ii) in Lemma 3.2 we know that the potential $p(\omega + v; \cdot)$ assumes its minimum on F at the point $x \in F \setminus F_1$. With the same objective as in (4.26) we define

$$c_2 := p(\omega + v; x) = \inf_{z \in F} p(\omega + v; z). \tag{4.83}$$

A comparison of the two potentials $p(\omega+v;\cdot)$ and $p(\omega;\cdot)$, and using (4.49), shows that

$$-c_2 \geqslant (1+\|v\|)\, c_0. \tag{4.84}$$

As in the analysis done in (4.27) through (4.30), we deduce from the assertions (i) and (ii) of Lemma 3.2 that there exist $\varepsilon_3 > 0$ and an open set V_3 with

$$F_1 \subseteq V_3, \qquad \overline{V_3} \cap L(x, \delta_0) = \varnothing, \tag{4.85}$$

and

$$p(\omega+v; z) \geqslant c_2 + \varepsilon_3 \qquad \text{for all} \quad z \in (F \cup \overline{V_3}) \backslash L(x, \delta_0). \tag{4.86}$$

From (4.55), (4.60), (4.74), (4.78), and (4.82) we know that

$$\frac{1}{m+n}\, v_{Q_n P_n} \overset{*}{\rightarrow} \omega + v \qquad \text{as} \quad n \rightarrow \infty, \quad n \in N, \tag{4.87}$$

and with (4.86) and the principle of descent (Lemma 2.1), as in (4.30), this implies that

$$\limsup_{n \rightarrow \infty,\, n \in N} \frac{1}{m+n} \log |(Q_n P_n)(z)| \leqslant -p(\omega+v; z) \leqslant -c_2 - \varepsilon_3 \tag{4.88}$$

for all $z \in (F \cup \overline{V_3}) \backslash L(x, \delta_0)$, and the inequality between the limsup and the last term in (4.88) holds uniformly for $z \in (F \cup \overline{V_3}) \backslash L(x, \delta_0)$. With the same techniques as those applied in (4.31) through (4.43) of the proof of Theorem 1.8 (first part), we then deduce from (4.83), (4.88), and the assertions proved in Lemma 3.2 that

$$\lim_{n \rightarrow \infty,\, n \in N} \frac{1}{m+n} \log \left| \oint_C (P_n Q_n f)(\zeta) \zeta^{m+n}\, d\zeta \right| = -c_2. \tag{4.89}$$

As in the proof of Theorem 1.8 (first part), the division of the integration path into $C_0 \backslash L(x, \delta_0)$ and $C_0 \cap L(x, \delta_0)$ is the basis for proving (4.89) here.

Because of (4.84) the limit (4.89) contradicts (4.81), which proves that the functions $(1/(m+n)) \log |H_n(z)|$ converge in capacity to the constant c_0 in the domain D. With this result we are close to the completion of our proof of (1.10).

Let $V \subseteq D \backslash \{\infty\}$ be an arbitrary compact set and $\tilde{\varepsilon} > 0$ a number with $\tilde{\varepsilon} < G(z)$ for all $z \in V$. A little differently from (4.60), we now define

$$\tilde{E}_n := \{ z \in V \mid |(f - [m/n])(z)| < (G(z) - \tilde{\varepsilon})^{m+n} \}. \tag{4.90}$$

There exists $\varepsilon > 0$ such that

$$\log \left| 1 - \frac{\tilde{\varepsilon}}{G(z)} \right| \leqslant -\varepsilon \qquad \text{for all} \quad z \in V \tag{4.91}$$

(compare with (4.61)). From error formula (4.46) and the definition of the functions f_n in (4.64) it follows that

$$\begin{aligned} &\frac{1}{m+n} \log \frac{|(f-[m/n])(z)|}{\mathrm{G}(z)^{m+n}} \\ &\quad = -\frac{2}{m+n} \log |Q_n(z)| - \log G(z) + \frac{m-n}{m+n} \log |z| + \frac{1}{m+n} \log - |H_n(z)| \\ &\quad = f_n(z) + \frac{m-n}{m+n} \log |z| + \left(\frac{1}{m+n} \log |H_n(z)| - c_0 \right). \end{aligned} \tag{4.92}$$

Since we know that all three terms on the right-hand side of (4.92) converge to 0 in capacity in the domain D, we know from Lemma 2.16 that also the left-hand side of (4.90) converges to 0 in capacity in D. Hence, the sets

$$E_n := \left\{ z \in V \,\middle|\, \frac{1}{m+n} \log \frac{|(f-[m/n])(z)}{G(z)^{m+n}} < -\varepsilon \right\}, \qquad n \in N, \tag{4.93}$$

satisfy

$$\operatorname{cap}(E_n) \to 0 \qquad \text{as} \quad n \to \infty, \qquad n \in N. \tag{4.94}$$

From (4.91) and the two definitions (4.90) and (4.93), it follows that $\tilde{E}_n \subseteq E_n$ for all $n \in N$ and therefore $\operatorname{cap}(\tilde{E}_n) \to 0$ as $n \to \infty$, $n \in N$, which proves limit (1.10) not only for the subsequence N, but also for the orginal sequence. ▮

Proofs of Theorems 1.1, 1.2, and 1.4. These three theorems are so closely connected that it is best to prove them together. Basically, the theorems are corollaries of the Nuttall–Pommerenke Theorem if the function f is single-valued in $\overline{\mathbb{C}} \backslash E$, and of Theorem 1.7 otherwise. In the latter case, knowledge about extremal domains, which has been summarized in the Theorems 1.3, 1.5, and 1.6, has to be used.

Let the function f satisfy Assumption 1.1, let $E \subseteq \overline{\mathbb{C}}$ denote the set of singularities of f, and let us assume, as in the proofs of Theorem 1.8 (first part) and Theorem 1.7, for the sake of simplicity that in the sequence $\{(m_j, n_j)\}_{j \in \mathbb{N}}$ of indices each n_j appears at most once. Thus, we can write, as before, n instead of n_j and $m = m(n)$ instead of m_j, $n \in N \subseteq \mathbb{N}$. It is assumed that $\{(m, n)\}$ satisfies (1.6), which is equivalent to the limit (4.12).

If the function f is single-valued in the domain $\overline{\mathbb{C}}\backslash E$, then it satisfies the assumptions of the Nuttall–Pommerenke Theorem. Condition (1.6) implies the assumption (1.5). Hence, we know that the sequence $[m/n]$, $n \in N$, converges to f in capacity in $\mathbb{C}$ as $n \to \infty$, $n \in N$. Of course, an essentially larger convergence domain is not possible, and Theorem 1.1 is proved.

From the convention made before (1.8), it follows that $g_D(z, \infty) = \infty$ for all $z \in D := \overline{\mathbb{C}}$, and consequently $G(z) = G_D(z) = 0$ for all $z \in \overline{\mathbb{C}}$. Hence, in the case of f being single-valued in $D\backslash E$, limit (1.9) follows from limit (1.4) in the Nuttall–Pommerenke Theorem. The limit (1.10) does not need to be proved since $G \equiv 0$. From Theorem 1.3 it immediately follows that in the special case under consideration the domain of the single-valued analytic continuation of f is $D = \overline{\mathbb{C}}\backslash E$. Therefore Theorem 1.4 is also verified in the special case.

It may be of interest to add that the proof of the Nuttall–Pommerenke Theorem is much simpler than the proofs of Theorems 1.7 and 1.8, which have to be used in the case of a function f with branch points. For the Nuttall–Pommerenke Theorem it is enough to prove a good estimate of the function H_n defined in (4.46), which can be done by an appropriately chosen integration path and a correspondingly chosen polynomial P in (4.48). Especially, no knowledge about the asymptotic distribution of the zeros of the denominator polynomials Q_n is necessary, which is the key to the proof of Theorem 1.7.

Next we assume that f is not single-valued in $\overline{\mathbb{C}}\backslash E$. Let D be the extremal domain of the single-valued analytic continuation of f, the unique existence of which follows from Theorem 1.3. Let F denote the complement of $\overline{\mathbb{C}}\backslash D$. From Theorem 1.5 we know that there exists a compact set F_0 with $\operatorname{cap}(F_0) = 0$ and that $F\backslash F_0$ is the union of disjoint, open, analytic Jordan arcs J_j, $j \in I$, i.e.,

$$F\backslash F_0 = \bigcup_{j \in I} J_j \,. \tag{4.95}$$

Since f is multi-valued in $\overline{\mathbb{C}}\backslash E$, and since from Theorem 1.5 we know that $\operatorname{cap}(F_0\backslash E) = 0$, it follows that $F\backslash F_0 \neq \emptyset$ or equivalently that $I \neq \emptyset$. With Theorem 1.6 we then deduce that D is a symmetric domain in the sense of Definition 1.3. The function f is locally analytic in $\overline{\mathbb{C}}\backslash E$, and consequently f has analytic continuations from both sides of J_j, and each jump function g_j, $j \in I$, introduced in (1.26), is analytic on J_j. Further we deduce that each $g_j \not\equiv 0$, $j \in I$. Indeed, if for some $j \in I$ we would have $g_j \equiv 0$, then J_j could be removed from F, and f would nevertheless be single-valued in the new domain $\tilde{D}$, but this contradicts the minimality of $\operatorname{cap}(\partial D) = \operatorname{cap}(F)$ in assertion (ii) of Theorem 1.3. Since all jump functions g_j, $j \in I$, are analytic and not identical to zero on J_j, it follows that all assumptions of Theorem

1.7 are satisfied, and we therefore know that the sequence of Padé approximants $[m/n]$, $n \in N$, converges to f in capacity in the extremal domain D. Further, it follows that the two limits (1.9) and (1.10) hold true with a function $G = G_D$ defined in the extremal domain D. To complete the proof of Theorems 1.1, 1.2, and 1.4 it only remains to show that there does not exist a domain $\tilde{D}$ essentially larger than D, in which the sequence $\{[m/n]\}$ converges in capacity to the analytic continuation of f.

Let $\tilde{D}$ be a domain with $D \subseteq \tilde{D}$ and $\operatorname{cap}(\tilde{D} \backslash D) > 0$. Since $\operatorname{cap}(F_0) = 0$, it follows from (1.13) in Theorem 1.5 that there exists an open subarc $\tilde{J} \subseteq F \backslash F_0$ with $\tilde{J} \subseteq \tilde{D}$. Let us assume that $\tilde{J} \subseteq J_j$ with $j \in I$. Since the jump function $g_j \not\equiv 0$, it follows that the analytic continuation of f into $\tilde{D}$ is multi-valued in a neighborhood of $\tilde{J}$. In [11] it has been shown that if a sequence converges in capacity in an subdomain of $\mathbb{C}$, then one can select an infinite subsequence that converges quasi everywhere in the same domain. Since $\tilde{J} \subseteq \tilde{D}$, the analytic continuation of f is multi-valued in a neighborhood of $\tilde{J}$ and therefore convergence in capacity of the sequence $\{[m/n]\}$ is not possible in a full neighborhood of $\tilde{J}$, and hence also not in the domain $\tilde{D}$. This completes the proof of Theorems 1.1, 1.2, and 1.4. ∎

Proof of Theorem 1.8 (Second Part). In the first part it has been proved that the first limit in (1.28) holds true under the assumptions of Theorem 1.7. It has been shown above in the second part of the proof of Theorems 1.1, 1.2, and 1.4 that if the function f satisfies Assumption 1.1 and if it is multi-valued in $\overline{\mathbb{C}} \backslash E$, then the extremal domain $D = D_f$ for single-valued analytic continuation of f, defined after Theorem 1.3, is a symmetric domain in the sense of Definition 1.3. From Theorem 1.4 we know that it is also the convergence domain of Theorem 1.1. Further it has been shown that in this case the assumptions of Theorem 1.7 are all fullfilled. Hence, it follows from the first part of the present proof that the first limit in (1.28) holds true under both types of assumptions, those of Theorem 1.7 and those of Theorem 1.1 if the function f is not single-valued in $\overline{\mathbb{C}} \backslash E$. Thus, only the second limit in (1.28) remains to be proved, which we shall do now. It will be deduced from the first limit in (1.28).

Let us again assume without loss of generality that in the sequence $\{(m_j, n_j)\}_{j \in \mathbb{N}}$ each n_j appears at most once. For the sake of simplicity we then can write n instead of n_j and $m = m(n)$ instead of m_j, $n \in N \subseteq \mathbb{N}$. As in Theorem 1.8 it is assumed that the sequence $\{(m, n)\}_{n \in N}$ satisfies limit (1.6) or equivalently limit (4.12).

Let Q_n and P_n denote the coprime denominator and numerator polynomials of the Padé approximant $[m/n]$, i.e.

$$[m/n](z) = \frac{P_n(z)}{Q_n(z)} = \frac{p_{mn}(1/z)}{q_{mn}(1/z)}, \qquad Q_n \in \mathcal{P}_n^*, \tag{4.96}$$

with p_{mn} and q_{mn} defined in (1.3). The polynomial Q_n has been introduced in (4.1). We have

$$\pi_{mn} = \nu_{Q_n}, \qquad \zeta_{mn} = \nu_{P_n}. \tag{4.97}$$

The first identity has already been used in (4.13). From the first limit in (1.29) we know that

$$\frac{1}{n}\,\nu_{Q_n} \xrightarrow{*} \omega \qquad \text{as} \quad n \to \infty,\ n \in N, \tag{4.98}$$

where $\omega = \omega_F$ is the equilibrium distribution on the set $F = \overline{\mathbb{C}} \backslash D$. By choosing an infinite subsequence, if necessary, and which we continue to denote by N, we can assume that the limit

$$\frac{1}{n}\,\nu_{P_n} \xrightarrow{*} \nu \qquad \text{as} \quad n \to \infty,\ n \in N, \tag{4.99}$$

exists. Because of limit (4.12) the measure ν is positive and $|\nu| < 1$. The proof of Theorem 1.8 is complete when we have shown that $\nu = \omega$.

In Theorems 1.1 and 1.7 we have proved that the sequence of Padé approximants $[m/n]$, $n \in N$, converges to f in capacity in the domain D. In [11] it has been shown that convergence in capacity implies convergence quasi everywhere for an infinite subsequence. Thus, we can select such a sequence, which we continue to denote by N, such that $[m/n](z) \to f(z)$ for quasi every $z \in D$. From (4.96) it then follows that

$$\lim_{n \to \infty,\, n \in N} \frac{1}{n} \log \left| \frac{P_n(z)}{Q_n(z)} \right| = \lim_{n \to \infty,\, n \in N} \left[c_n + p\left(\frac{1}{n}\,\nu_{Q_n} - \frac{1}{n}\,\nu_{P_n}; z \right) \right] = 0 \tag{4.100}$$

for quasi every $z \in D$. The constants $c_n \in \mathbb{R}$ in (4.100) are defined by

$$\frac{1}{n} \log |P_n(z)| = c_n - p\left(\frac{1}{n}\,\nu_{P_n}; z \right). \tag{4.101}$$

Since the logarithmic potentials are normalized by (2.1) and (2.3) in such a way that they always are finite quasi everywhere in $\mathbb{C}$, it follows from the limits (4.98), (4.99), (4.100), and the Lower Envelope Theorem (Lemma 2.2) that the limit

$$\lim_{n \to \infty,\, n \in N} c_n =: c_0 \tag{4.102}$$

exists and that

$$p(\omega - \nu; z) = -c_0 \tag{4.103}$$

holds for quasi every $z \in D$. Since $\text{supp}(\omega) \subseteq F$, the potential $p(\omega - \nu; \cdot)$ is subharmonic in D. This implies that (4.103) has to hold for all $z \in D$. The potential $p(\omega - \nu; \cdot)$ is continued in the fine topology. The ordinary boundary of a domain is also the fine boundary of this domain. From Theorem 1.5 or from Definition 1.3 we know that $\overline{\mathbb{D}} = \overline{\mathbb{C}}$. Hence, it follows from the fine continuity of $p(\omega - \nu; \cdot)$ that (4.103) holds for all $z \in \overline{\mathbb{C}}$, which implies

$$\nu = \omega \qquad \text{and} \qquad c_0 = 0, \tag{4.104}$$

and the proof is completed.

Proof of Theorem 1.9. It has been shown in the Proof of Theorems 1.1, 1.2, and 1.4 that it makes no difference whether one starts from the assumptions of Theorem 1.7 or Theorem 1.1 if in the latter case the function f has branch points. Therefore, we can use all results from the proofs of both theorems. Especially, the error formula (4.46) and elements of the proof after (4.68) shall be used, where it has been proved that the sequence $((1/(m+n)) \log |H_n(z)|$ converges in capacity to the constant c_0 in the domain D. As before we will write n instead of n_j and $m = m(n)$ instead of m_j, $n \in N \subseteq \mathbb{N}$. Let o_n denote the order of the zero of the error function $e_n(z) := f(z) - [m/n](z)$ at the point $z = \infty$. After (4.68) it has been shown that the measures ν_n introduced in (4.68) converge to zero for any subdomain D_1. This implies with the error formula (4.46) that

$$\lim_{n \to \infty,\, n \in N} \frac{1}{n} |o_n - 2 \deg(Q_n)| = 0. \tag{4.105}$$

With (4.12) and the first limit in (1.29) of Theorem 1.8, it then follows that

$$\lim_{n \to \infty,\, n \in N} \frac{1}{n} |m + n - o_n| = 0, \tag{4.106}$$

which proves limit (1.30).

Since

$$\deg([m/n]) = \max(\deg(Q_n), \deg(P_n)), \tag{4.107}$$

limit (1.31) follows from both limits in (1.29) of Theorem 1.8. ∎

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